Title: The isometrization of groups of homeomorphism

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Abstract: Let X be a Hausdorff space. A *gauge (pseudometric)* on X is a function $\rho : X \times X \to [0,\infty)$ satisfying: (1) $x = y \Rightarrow \rho(x,y) = 0$, (2) $\rho(x,y) = \rho(y,x)$, (3) $\rho(x,z) \leq \rho(x,y) + \rho(y,z)$, and (4) every $\mathcal{N}_\rho(x,\epsilon) = \{ y \in X : \rho(x,y) < \epsilon \}$ is an open subset of X. A set $\mathcal{P}$ of gauges on X is a *gauge structure* on X if $\{ \mathcal{N}_\rho(x,\epsilon) : \rho \in \mathcal{P}, x \in X, \epsilon > 0 \}$ is a subbasis for the topology on X. A gauge ρ on X is *proper* if every $\mathcal{N}_\rho(x,\epsilon)$ has compact closure, and a gauge structure is *proper* if all of its elements are proper. **Note:** Since X is Hausdorff, then: $\mathcal{P}$ is a gauge structure on X $\Rightarrow \forall x \neq y \in X, \exists \rho \in \mathcal{P}$ such that $\rho(x,y) > 0$. Hence, if $\mathcal{P} = \{\rho\}$ is a gauge structure on X, then ρ is a metric on X that induces the given topology.

Let G be a group of homeomorphisms of a Hausdorff space X.

A gauge structure $\mathcal{P}$ on X is *G-invariant* (and G is a *$\mathcal{P}$-isometry group)* if $\forall g \in G, \forall \rho \in \mathcal{P}$ and $\forall x, y \in X$, $\rho(g(x),g(y)) = \rho(x,y)$. G is *(properly) isometrizable* if there is a G-invariant (proper) gauge structure on X (i.e., there is a (proper) gauge structure $\mathcal{P}$ on X which makes G a $\mathcal{P}$-isometry group). G is *equiregular* if for every $x \in X$ and every open neighborhood U of x in X there is an open neighborhood V of x in X such that $cl(V) \subset U$ and every $y \in X$ has an open neighborhood N_y with the property that for every $g \in G$, if $g(N_y) \cap cl(V) \neq \emptyset$, then $g(N_y) \subset U$. G is *nearly proper* if for all compact subsets A and $B \subset X$, $\cup\{ g(A) : g \in G \text{ and } g(A) \cap B \neq \emptyset \}$ has compact closure.

The Isometrization Theorem. If X is a Hausdorff space and G\X is a paracompact regular space, then: G is isometrizable if and only if G is equiregular.

The Proper Isometrization Theorem. If X is a locally compact σ-compact Hausdorff space and G\X is a regular space, then: G is properly isometrizable if and only if G is equiregular and nearly proper.

G is *singly (properly) isometrizable* if there is a one-element G-invariant (proper) gauge structure on X (i.e., X is metrizable by a (proper) metric ρ which makes each element of G a ρ-isometry).

Corollary. If X is a separable metrizable space, G\X is a paracompact regular space and G is equiregular, then G is singly isometrizable.

Corollary. If X is a locally compact σ-compact metrizable space, G\X is a regular space, and G is equiregular and nearly proper, then G is singly properly isometrizable.

Question. If X is a non-separable metrizable space, G\X is a paracompact regular space and G is equiregular, must G be singly isometrizable?

The Proper Isometrization Theorem generalizes results of Abel-Manoussos-Noskov (2011) and Antonyan-de Neymet (2003).