System of Differential Equations over Quaternion Algebra

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Abstract. In order to study homogeneous system of linear differential equations, I considered vector space over division D-algebra and the theory of eigenvalues in non commutative division D-algebra. Since product in algebra is non-commutative, I considered two forms of product of matrices (section 2) and two forms of eigenvalues (section 4). In sections 5, 6, 7, I considered solving of homogenius system of differential equations.

Contents

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http://AleksKleyn.dyndns-home.com:4080/ http://arxiv.org/a/kleyn_a_1.
http://sites.google.com/site/AleksKleyn/ http://AleksKleyn.blogspot.com/.

1. Convention

Convention 1.1. *Let A be free algebra with finite or countable basis. Considering expansion of element of algebra A relative basis $\overline{\overline{e}}$ we use the same root letter to denote this element and its coordinates. In expression a^2, it is not clear whether this is component of expansion of element a relative basis, or this is operation $a^2 = aa$. To make text clearer we use separate color for index of element of algebra. For instance,*

$$a = a^i e_i$$

□

Convention 1.2. *We will use Einstein summation convention in which repeated index (one above and one below) implies summation with respect to repeated index. In this case we assume that we know the set of summation index and do not use summation symbol*

$$c^i v_i = \sum_{i \in I} c^i v_i$$

If needed to clearly show a set of indices, I will do it. □

2. Biring

Let A be associative division algebra over commutative ring D. We also will say that A is associative D-algebra.

Left or right module V over division D-algebra A is called A-vector space.

According to the custom the product of matrices a and b is defined as product of rows of the matrix a and columns of the matrix b.

Example 2.1. *Let $\overline{\overline{e}}$ be basis of right vector space V. We represent the basis $\overline{\overline{e}}$ as row of matrix*

$$e = \begin{pmatrix} e_1 & \dots & e_n \end{pmatrix}$$

We represent coordinates of vector v as vector column

$$v = \begin{pmatrix} v^1 \\ \dots \\ v^n \end{pmatrix}$$

Therefore, we can represent the vector v as product of matrices

$$v = \begin{pmatrix} e_1 & \dots & e_n \end{pmatrix} \begin{pmatrix} v^1 \\ \dots \\ v^n \end{pmatrix} = e_i v^i$$

□

Example 2.2. *Let $\overline{\overline{e}}$ be basis of left vector space V. We represent the basis $\overline{\overline{e}}$ as row of matrix*

$$e = \begin{pmatrix} e_1 & \dots & e_n \end{pmatrix}$$

We represent coordinates of vector v as vector column

$$v=\begin{pmatrix}v^1\\ ...\\ v^n\end{pmatrix}$$

However, we cannot represent the vector

$$v=v^ie_i$$

as product of matrices

$$v=\begin{pmatrix}v^1\\ ...\\ v^n\end{pmatrix}\qquad e=\begin{pmatrix}e_1 & ... & e_n\end{pmatrix}$$

because this product is not defined. □

From examples 2.1, 2.2, it follows that we cannot confine ourselves to traditional product of matrices and we need to define two products of matrices. To distinguish between these products we introduced a new notation.

Definition 2.3. *Let the nubmer of columns of the matrix a equal the number of rows of the matrix b.* ${}_*{}^*$**-product** *of matrices a and b has form*

$$\left\{\begin{array}{r}a_*{}^*b=\left(a_k^ib_j^k\right)\\ (a_*{}^*b)_j^i=a_k^ib_j^k\end{array}\right. \tag{2.1}$$

$$\begin{aligned}\begin{pmatrix}a_1^1 & ... & a_p^1\\ ... & ... & ...\\ a_1^n & ... & a_p^n\end{pmatrix}{}_*{}^*\begin{pmatrix}b_1^1 & ... & b_m^1\\ ... & ... & ...\\ b_1^p & ... & b_m^p\end{pmatrix}&=\begin{pmatrix}a_k^1b_1^k & ... & a_k^1b_m^k\\ ... & ... & ...\\ a_k^nb_1^k & ... & a_k^nb_m^k\end{pmatrix}\\ &=\begin{pmatrix}(a_*{}^*b)_1^1 & ... & (a_*{}^*b)_m^1\\ ... & ... & ...\\ (a_*{}^*b)_1^n & ... & (a_*{}^*b)_m^n\end{pmatrix}\end{aligned} \tag{2.2}$$

${}_*{}^*$*-product can be expressed as product of a row of the matrix a over a column of the matrix b.* □

Definition 2.4. *Let the nubmer of rows of the matrix a equal the number of columns of the matrix b.* ${}^*{}_*$**-product** *of matrices a and b has form*

$$\left\{\begin{array}{r}a^*{}_*b=\left(a_i^kb_k^j\right)\\ (a^*{}_*b)_j^i=a_i^kb_k^j\end{array}\right. \tag{2.3}$$

$$\begin{pmatrix} a_1^1 & ... & a_m^1 \\ ... & ... & ... \\ a_1^p & ... & a_m^p \end{pmatrix} {}^*{}_* \begin{pmatrix} b_1^1 & ... & b_p^1 \\ ... & ... & ... \\ b_1^n & ... & b_p^n \end{pmatrix} = \begin{pmatrix} a_1^k b_k^1 & ... & a_m^k b_k^1 \\ ... & ... & ... \\ a_1^k b_k^n & ... & a_m^k b_k^n \end{pmatrix} = \begin{pmatrix} (a_*{}^*b)_1^1 & ... & (a_*{}^*b)_m^1 \\ ... & ... & ... \\ (a_*{}^*b)_1^n & ... & (a_*{}^*b)_m^n \end{pmatrix} \tag{2.4}$$

${}^*{}_*$-product can be expressed as product of a column of the matrix a over a row of the matrix b. □

Remark 2.5. *We will use symbol ${}_*{}^*$- or ${}^*{}_*$- in name of properties of each product and in the notation. We can read the symbol ${}_*{}^*$ as rc-product (product of row over column) and the symbol ${}^*{}_*$ as cr-product (product of column over row). In order to keep this notation consistent with the existing one we assume that we have in mind ${}_*{}^*$-product when no clear notation is present.* □

Definition 2.6. *Matrix $\delta = (\delta_j^i)$ is identity for both products.* □

Definition 2.7. *We introduce* ${}_*{}^*$**-power** *of matrix a using recursive definition*

$$a^{0_*{}^*} = \delta \tag{2.5}$$

$$a^{n_*{}^*} = a^{n-1_*{}^*}{}_*{}^*a \tag{2.6}$$

□

Theorem 2.8.

$$a^{1_*{}^*} = a \tag{2.7}$$

Definition 2.9. *The matrix $a^{-1_*{}^*}$ is* ${}_*{}^*$**-inverse element** *of the matrix a if*

$$a_*{}^*a^{-1_*{}^*} = \delta \tag{2.8}$$

Matrix a is called ${}_*{}^*$-regular, if there exists ${}_*{}^*$-inverse matrix. □

Definition 2.10. *The matrix $a^{-1^*{}_*}$ is* ${}^*{}_*$**-inverse element** *of the matrix a if*

$$a^*{}_*a^{-1^*{}_*} = \delta \tag{2.9}$$

Matrix a is called ${}^*{}_*$-regular, if there exists ${}^*{}_*$-inverse matrix. □

3. Quasideterminant

According to [1], page 3 we do not have an appropriate definition of a determinant for a division algebra.[1] However, we can define a quasideterminant which finally gives a similar picture. In definition 3.1, I follow the definition [1]-1.2.2.

[1]Professor Kyrchei uses double determinant (see the definition in the section [4]-2.2) to solve system of linear equations in quaternion algebra and to solve eigenvalues problem (see the section [4]-2.5). I confine myself by consideration of quasideterminant, because I am interested in a wider set of algebras.

[4] Ivan Kyrchei, Linear differential systems over the quaternion skew field, eprint arXiv:1812.03397 (2018)

[1] I. Gelfand, S. Gelfand, V. Retakh, R. Wilson, Quasideterminants, eprint arXiv:math.QA/0208146 (2002)

Definition 3.1. $\binom{j}{i}$**-${}_*{}^*$-quasideterminant** *of $n\times n$ matrix a is formal expression*

$$\det({}_*{}^*)^j_i\, a = ((a^{-1_*{}^*})^i_j)^{-1} \tag{3.1}$$

We consider $\binom{j}{i}$-${}_*{}^*$-quasideterminant as an entry of the matrix

$$\begin{aligned}\det({}_*{}^*)\, a &= \begin{pmatrix}\det({}_*{}^*)^1_1\, a & ... & \det({}_*{}^*)^1_n\, a\\ ... & ... & ...\\ \det({}_*{}^*)^n_1\, a & ... & \det({}_*{}^*)^n_n\, a\end{pmatrix}\\ &= \begin{pmatrix}((a^{-1_*{}^*})^1_1)^{-1} & ... & ((a^{-1_*{}^*})^n_1)^{-1}\\ ... & ... & ...\\ ((a^{-1_*{}^*})^1_n)^{-1} & ... & ((a^{-1_*{}^*})^n_n)^{-1}\end{pmatrix}\end{aligned} \tag{3.2}$$

which is called ${}_*{}^*$**-quasideterminant**. □

Theorem 3.2. *Consider matrix*

$$\begin{pmatrix}a^1_1 & a^1_2\\ a^2_1 & a^2_2\end{pmatrix}$$

Then

$$\det({}_*{}^*)a = \begin{pmatrix}a^1_1 - a^1_2(a^2_2)^{-1}a^2_1 & a^1_2 - a^1_1(a^2_1)^{-1}a^2_2\\ a^2_1 - a^2_2(a^1_2)^{-1}a^1_1 & a^2_2 - a^2_1(a^1_1)^{-1}a^1_2\end{pmatrix} \tag{3.3}$$

$$\det({}^*{}_*)a = \begin{pmatrix}a^1_1 - a^2_1(a^2_2)^{-1}a^1_2 & a^1_2 - a^2_2(a^2_1)^{-1}a^1_1\\ a^2_1 - a^1_1(a^1_2)^{-1}a^2_2 & a^2_2 - a^1_2(a^1_1)^{-1}a^2_1\end{pmatrix} \tag{3.4}$$

$$a^{-1_*{}^*} = \begin{pmatrix}(a^1_1 - a^2_1(a^2_2)^{-1}a^1_2)^{-1} & (a^2_1 - a^1_1(a^1_2)^{-1}a^2_2)^{-1}\\ (a^1_2 - a^2_2(a^2_1)^{-1}a^1_1)^{-1} & (a^2_2 - a^1_2(a^1_1)^{-1}a^2_1)^{-1}\end{pmatrix} \tag{3.5}$$

4. Eigenvalue of Matrix

Let a be $n\times n$ matrix of A-numbers and E be $n\times n$ unit matrix.

Definition 4.1. *A-number b is called* ${}_*{}^*$**-eigenvalue** *of the matrix a if the matrix $a-bE$ is ${}_*{}^*$-singular matrix.* □

Definition 4.2. *Let A-number b be ${}_*{}^*$-eigenvalue of the matrix a. A-column v is called* ${}_*{}^*$**-eigencolumn** *of matrix a corresponding to ${}_*{}^*$-eigenvalue b, if the following equality is true*

$$a{}_*{}^*v = bv \tag{4.1}$$

□

Definition 4.3. *Let A-number b be ${}_*{}^*$-eigenvalue of the matrix a. A-row v is called* **${}_*{}^*$-eigenrow** *of matrix a corresponding to ${}_*{}^*$-eigenvalue b, if the following equality is true*

$$v{}_*{}^*a = vb \tag{4.2}$$

□

Definition 4.4. *A-number b is called* **${}^*{}_*$-eigenvalue** *of the matrix a if the matrix $a - bE$ is ${}^*{}_*$-singular matrix.* □

Definition 4.5. *Let A-number b be ${}^*{}_*$-eigenvalue of the matrix a. A-column v is called* **${}^*{}_*$-eigencolumn** *of matrix a corresponding to ${}^*{}_*$-eigenvalue b, if the following equality is true*

$$v{}^*{}_*a = vb \tag{4.3}$$

□

Definition 4.6. *Let A-number b be ${}^*{}_*$-eigenvalue of the matrix a. A-row v is called* **${}^*{}_*$-eigenrow** *of matrix a corresponding to ${}^*{}_*$-eigenvalue b, if the following equality is true*

$$a{}^*{}_*v = bv \tag{4.4}$$

□

5. Differential Equation $\dfrac{dx}{dt} = ax$

Theorem 5.1. *Let A be non-commutative D-algebra. For any $b \in A$, there exists subalgebra $Z(A,b)$ of D-algebra A such that*

$$c \in Z(A,b) \Rightarrow cb = bc \tag{5.1}$$

D-algebra $Z(A,b)$ is called **center of A-number** *b.*

Theorem 5.2. *Since $a \in Z(A,b)$, then $b \in Z(A,a)$.*

Definition 5.3. *The map*

$$y = e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n \tag{5.2}$$

is called exponent. □

Theorem 5.4. *Let A be Banach D-algebra and $a \in A$. The map*

$$f : t \in R \to e^{at} \in A$$

has the following Taylor series decomposition

$$e^{at} = \sum_{n=0}^{\infty} \frac{1}{n!} a^n t^n \tag{5.3}$$

Proof. The theorem follows from the statement $t \in Z(A,a)$. □

The theorem 5.5 is important for consideration of system of differential equations.

Theorem 5.5. *Let A be Banach associative D-algebra and a, $c \in A$. The condition*

$$c \in Z(A, a) \tag{5.4}$$

implies that

$$e^{at}c = ce^{at} \tag{5.5}$$

Theorem 5.6. *Let A be Banach D-algebra and $a \in A$. Then*

$$e^{at}a = ae^{at} \tag{5.6}$$

PROOF. The theorem follows from the theorem 5.5 if we set $c = a$. □

Theorem 5.7. *Let*

$$f : R \to A$$

be a map of real field R into Banach D-algebra A. The derivative of order n of the map f is the map

$$t \in R \to \frac{d^n f(t)}{dt^n} \in A$$

Theorem 5.8. *Let A be Banach D-algebra and $a \in A$. The derivative of order n of the map*

$$f : t \in R \to e^{at} \in A$$

has the following form

$$\frac{d^n e^{at}}{dt^n} = e^{at}a^n = a^n e^{at} \tag{5.7}$$

Theorem 5.9. *Let A be Banach D-algebra and $a \in A$. For any A-number c, the map*

$$x = e^{at}c \tag{5.8}$$

is solution of the differential equation

$$\frac{dx}{dt} = ax \tag{5.9}$$

The set of solutions of the differential equation (5.9) is right A-vector space

$$e^{at}A \subset \mathcal{R}A^R$$

generated by the map $x = e^{at}$.

PROOF. The equality

$$\frac{dx}{dt} = \frac{de^{at}c}{dt} = \frac{de^{at}}{dt}c = ae^{at}c = ax$$

follows from the theorem 5.8.

To the right of the exponent, we wrote an arbitrary constant on which the solution depends. To answer the question whether we can write a constant to the left of the exponent, we consider the lemma 5.10.

Lemma 5.10. *Let A be Banach D-algebra and $a \in A$. For any A-numbers c_1, c_2, the map*

$$x = c_1 e^{at} c_2 \tag{5.10}$$

is solution of the differential equation (5.9) iff $c_1 \in Z(A, a)$.

Proof. The equality

$$\frac{dx}{dt}=\frac{dc_1e^{at}c_2}{dt}=c_1\frac{de^{at}}{dt}c_2=c_1ae^{at}c_2$$

follows from the theorem 5.8. If $c_1\notin Z(A,a)$, then the condition

$$c_1a=ac_1 \tag{5.11}$$

is not true and the map (5.10) is not a solution of the differential equation (5.9). ⊙

According to the theorem 5.5, if $c_1\in Z(A,a)$, then the map (5.10) gets form

$$x=c_1e^{at}c_2=e^{at}c_1c_2 \tag{5.12}$$

and is the map of the form (5.8).

Therefore, the set of solutions (5.8) is right A-vector space. □

Theorem 5.11. *Let A be Banach D-algebra and $a\in A$. For any A-number c, the map*

$$x=ce^{at} \tag{5.13}$$

is solution of the differential equation

$$\frac{dx}{dt}=xa \tag{5.14}$$

The set of solutions of the differential equation (5.14) is left A-vector space

$$Ae^{at}\subset\mathcal{L}A^R$$

generated by the map $x=e^{at}$.

6. Differential Equation $\frac{dx}{dt}=a_*{}^*x$

Let A be Banach division D-algebra. The system of differential equations

$$\begin{aligned}\frac{dx^1}{dt}&=a^1_1x^1+...+a^1_nx^n\\ &......\\ \frac{dx^n}{dt}&=a^n_1x^1+...+a^n_nx^n\end{aligned} \tag{6.1}$$

where $a^i_j\in A$ and $x^i:R\to A$ is A-valued function of real variable, is called homogeneous system of linear differential equations.

Let

$$x=\begin{pmatrix}x^1\\ ...\\ x^n\end{pmatrix}\qquad\frac{dx}{dt}=\begin{pmatrix}\frac{dx^1}{dt}\\ ...\\ \frac{dx^n}{dt}\end{pmatrix}$$

$$a=\begin{pmatrix}a^1_1&...&a^1_n\\ ...&...&...\\ a^n_1&...&a^n_n\end{pmatrix}$$

Then we can write system of differential equations (6.1) in matrix form

$$\frac{dx}{dt}=a_*{}^*x \tag{6.2}$$

6.1. **Solution as exponent** $x=e^{bt}c$ **.** We will look for a solution of the system of differential equations (6.2) in the form of an exponent

$$x=e^{bt}c=\begin{pmatrix}e^{bt}c^{1}\\ \dots\\ e^{bt}c^{n}\end{pmatrix}\qquad c=\begin{pmatrix}c^{1}\\ \dots\\ c^{n}\end{pmatrix}\tag{6.3}$$

Theorem 6.1. *Let b be ${}_*{}^*$-eigenvalue of the matrix a. The condition*

$$b\in\bigcap_{i=1}^{n}\bigcap_{j=1}^{n}Z(A,a_j^i)\tag{6.4}$$

implies that the matrix of maps (6.3) is solution of the system of differential equations (6.2) for ${}_*{}^*$-eigencolumn c.

PROOF. According to the theorem 5.8, the equality

$$\frac{dx}{dt}=\frac{de^{bt}}{dt}c=e^{bt}bc=a_*{}^*x=a_*{}^*(e^{bt}c)\tag{6.5}$$

follows from equalities (6.2), (6.3). According to the theorem 5.5, the equality

$$e^{bt}bc=e^{bt}(a_*{}^*c)\tag{6.6}$$

follows from the equality (6.5) and from the statement (6.4). Since the expression e^{bt}, in general, is different from 0, the equality

$$a_*{}^*c=bc$$

follows from the equality (6.6). According to the definition 4.1, A-number b is ${}_*{}^*$-eigenvalue of the matrix a and the matrix c is ${}_*{}^*$-eigencolumn of matrix a corresponding to ${}_*{}^*$-eigenvalue b. □

Theorem 6.2. *Let b be ${}_*{}^*$-eigenvalue of the matrix a and do not satisfy to the condition (6.4). If entries of ${}_*{}^*$-eigencolumn c satisfy to the condition*

$$c^i=c_1^ip\quad p\in A\quad p\neq 0\tag{6.7}$$

$$c_1^i\in Z(A,b)\qquad i=1,...,n\tag{6.8}$$

then the matrix of maps (6.3) is solution of the system of differential equations (6.2).

PROOF. According to the theorem 5.8, the equality

$$\frac{dx}{dt}=\frac{de^{bt}}{dt}c=be^{bt}c_1p=a_*{}^*x=a_*{}^*(e^{bt}c_1)p\tag{6.9}$$

follows from equalities (6.2), (6.3), (6.7). According to the theorem 5.5, the equality

$$bc_1e^{bt}=(a_*{}^*c_1)e^{bt}\tag{6.10}$$

follows from the equality (6.9) and from statements (6.7), (6.8). Since the expression e^{bt}, in general, is different from 0, the equality

$$a_*{}^*c_1=bc_1$$

follows from the equality (6.10). According to the definition 4.1, A-number b is ${}_*{}^*$-eigenvalue of the matrix a and the matrix c_1 is ${}_*{}^*$-eigencolumn of matrix a

corresponding to ${}_*{}^*$-eigenvalue b. According to the theorem 13.9 the matrix c is ${}_*{}^*$-eigencolumn of matrix a corresponding to ${}_*{}^*$-eigenvalue b. □

Let ${}_*{}^*$-eigenvalue b does not satisfy to the condition (6.4). Let entries of ${}_*{}^*$-eigencolumn c do not satisfy to the condition (6.7), (6.8). Then the matrix of maps (6.3) is not solution of the system of differential equations (6.2).

Theorem 6.3. *Let $ev(a_*{}^*x)$ be the set of ${}_*{}^*$-eigenvalue of the matrix a for which there is a solution of the system of differential equations (6.2). Let $b \in ev(a_*{}^*x)$. The set $V(a_*{}^*x, b)$ of solutions (6.3) of the system of differential equations (6.2) is right A-vector space of columns.*

6.2. **Solution as exponent $x = ce^{bt}$.** In the paper [5], on page 35, authors suggest to consider solution as exponent

$$x = ce^{bt} \tag{6.11}$$

The equality

$$a_*{}^*(ce^{bt}) = cbe^{bt} \tag{6.12}$$

follows from equalities (6.2), (6.11). Since the expression e^{bt}, in general, is different from 0, the equality

$$a_*{}^*c = cb \tag{6.13}$$

follows from the equality (6.12).

[4] Ivan Kyrchei, Linear differential systems over the quaternion skew field, eprint arXiv:1812.03397 (2018)

[5] Kit Ian Kou, Yong-Hui Xia. Linear Quaternion Differential Equations: Basic Theory and Fundamental Results. eprint arXiv:1510.02224 (2017)

Based on the equality (6.13), authors of papers [4], [5] introduce the definition of right eigenvalue defined by the equality (6.13) versus left eigenvalue which we define by the equality

$$a_*{}^*c = bc$$

According to the lemma 5.10, if the matrix of maps (6.11) is a solution of the system of differential equations (6.2), then vector c satisfies to the condition

$$c^i \in Z(A, b) \qquad i = 1, ..., n$$

In such case, we can consider matrix of maps

$$x = e^{bt}c$$

instead of the matrix of maps (6.11) and we do not need to consider the definition of right eigenvalue.

6.3. **Method of successive differentiation.**

Theorem 6.4. *Differentiating one after another system of differential equations (6.2) we get the chain of systems of differential equations*

$$\frac{d^n x}{dt^n} = a^{n_*{}^*}{}_*{}^*x \tag{6.14}$$

Theorem 6.5. *The solution of the system of differential equations* (6.2) *with initial condition*

$$t=0 \qquad x=\begin{pmatrix}x^{1}\\ ...\\ x^{n}\end{pmatrix}=c=\begin{pmatrix}c^{1}\\ ...\\ c^{n}\end{pmatrix}$$

has the following form

$$x=e^{ta_*{}^*}{}_*{}^*c \tag{6.15}$$

7. Differential Equation $\frac{dx}{dt}=x^*{}_*a$

Let A be Banach division D-algebra. The system of differential equations

$$\begin{aligned}\frac{dx^{1}}{dt}&=x^{1}a^{1}_{1}+...+x^{n}a^{1}_{n}\\ &......\\ \frac{dx^{n}}{dt}&=x^{1}a^{n}_{1}+...+x^{n}a^{n}_{n}\end{aligned} \tag{7.1}$$

where $a^{i}_{j}\in A$ and $x^{i}:R\to A$ is A-valued function of real variable, is called homogeneous system of linear differential equations.

Let

$$x=\begin{pmatrix}x^{1}\\ ...\\ x^{n}\end{pmatrix}\qquad \frac{dx}{dt}=\begin{pmatrix}\frac{dx^{1}}{dt}\\ ...\\ \frac{dx^{n}}{dt}\end{pmatrix}$$

$$a=\begin{pmatrix}a^{1}_{1}&...&a^{1}_{n}\\ ...&...&...\\ a^{n}_{1}&...&a^{n}_{n}\end{pmatrix}$$

Then we can write system of differential equations (7.1) in matrix form

$$\frac{dx}{dt}=x^*{}_*a \tag{7.2}$$

We will look for a solution of the system of differential equations (7.2) in the form of an exponent

$$x=ce^{bt}=\begin{pmatrix}c^{1}e^{bt}\\ ...\\ c^{n}e^{bt}\end{pmatrix}\qquad c=\begin{pmatrix}c^{1}\\ ...\\ c^{n}\end{pmatrix} \tag{7.3}$$

Theorem 7.1. *Let b be ${}^*{}_*$-eigenvalue of the matrix a. The condition*

$$b\in\bigcap_{i=1}^{n}\bigcap_{j=1}^{n}Z(A,a^{i}_{j}) \tag{7.4}$$

implies that the matrix of maps (7.3) *is solution of the system of differential equations* (7.2) *for ${}^*{}_*$-eigencolumn c.*

Proof. According to the theorem 5.8, the equality

$$\frac{dx}{dt}=x^*{}_*a \tag{7.5}$$

follows from equalities (7.2), (7.3). According to the theorem 5.5, the equality

$$cbe^{bt}=(c^*{}_*a)e^{bt} \tag{7.6}$$

follows from the equality (7.5) and from the statement (7.4). Since the expression e^{bt}, in general, is different from 0, the equality

$$c^*{}_*a=cb$$

follows from the equality (7.6). According to the definition 4.4, A-number b is ${}^*{}_*$-eigenvalue of the matrix a and the matrix c is ${}^*{}_*$-eigencolumn of matrix a corresponding to ${}^*{}_*$-eigenvalue b. □

Theorem 7.2. *Let b be ${}^*{}_*$-eigenvalue of the matrix a and do not satisfy to the condition (7.4). If entries of ${}^*{}_*$-eigencolumn c satisfy to the condition*

$$c^i=pc_1^i \quad p\in A \quad p\ne 0 \tag{7.7}$$

$$c_1^i\in Z(A,b) \qquad i=1,...,n \tag{7.8}$$

then the matrix of maps (7.3) is solution of the system of differential equations (7.2).

Proof. According to the theorem 5.8, the equality

$$\frac{dx}{dt}=x^*{}_*a \tag{7.9}$$

follows from equalities (7.2), (7.3), (7.7). According to the theorem 5.5, the equality

$$e^{bt}c_1b=e^{bt}(c_1{}^*{}_*a) \tag{7.10}$$

follows from the equality (7.9) and from statements (7.7), (7.8). Since the expression e^{bt}, in general, is different from 0, the equality

$$c_1{}^*{}_*a=c_1b$$

follows from the equality (7.10). According to the definition 4.4, A-number b is ${}^*{}_*$-eigenvalue of the matrix a and the matrix c_1 is ${}^*{}_*$-eigencolumn of matrix a corresponding to ${}^*{}_*$-eigenvalue b. According to the theorem 13.10 the matrix c is ${}^*{}_*$-eigencolumn of matrix a corresponding to ${}^*{}_*$-eigenvalue b. □

Let ${}^*{}_*$-eigenvalue b does not satisfy to the condition (7.4). Let entries of ${}^*{}_*$-eigencolumn c do not satisfy to the condition (7.7), (7.8). Then the matrix of maps (7.3) is not solution of the system of differential equations (7.2).

Theorem 7.3. *Let $ev(x^*{}_*a)$ be the set of ${}^*{}_*$-eigenvalue of the matrix a for which there is a solution of the system of differential equations (7.2). Let $b\in ev(x^*{}_*a)$. The set $V(x^*{}_*a,b)$ of solutions (7.3) of the system of differential equations (7.2) is left A-vector space of columns.*

8. Elliptical Trigonometry

Consider the system of differential equations

$$\begin{aligned}\frac{dx^1}{dt}&=x^2\\ \frac{dx^2}{dt}&=-x^1\end{aligned}\tag{8.1}$$

The matrix a has form

$$a=\begin{pmatrix}0&1\\-1&0\end{pmatrix}$$

Since entries of matrix a are real numbers, then the equation to find eigenvalue is

$$\begin{vmatrix}-b&1\\-1&-b\end{vmatrix}=b^2+1=0\tag{8.2}$$

It is evident that roots of the equation (8.2) depend on choice of D-algebra A.

Theorem 8.1. *In quaternion algebra, the equation* (8.2) *has infinitely many roots*

$$b=b^1i+b^2j+b^3k$$

such that

$$(b^1)^2+(b^2)^2+(b^3)^2=1$$

According to the theorem 6.1, the solution of the system of differential equations corresponding to eigennumber b, has form

$$x=e^{bt}\begin{pmatrix}c_b^1\\c_b^2\end{pmatrix}\tag{8.3}$$

where H-column

$$\begin{pmatrix}c_b^1\\c_b^2\end{pmatrix}$$

is eigenvector of the matrix a. Coefficients c_b^1, c_b^2 which correspond to given eigenvalue b, satisfy to the equation

$$-bc_b^1+c_b^2=0$$

Therefore, corresponding solution of the system of differential equations (8.1) has form

$$x^1=e^{bt}\qquad x^2=e^{bt}b\tag{8.4}$$

If we want to find the solution of the system of differential equations (8.1) which satisfies to initial condition

$$t=0\quad x^1=0\quad x^2=1$$

then first impression is that we have too many choices.

Linear combination of two solutions of the system of differential equations (8.1) is solution of the system of differential equations (8.1). We will start from consideration of linear combination of two solutions of the form (8.4) because, in such case, constants of linear combination are unique.

Thus, we search solution of the form

$$\begin{aligned} x^{1} &= e^{b_1 t}C_1 + e^{b_2 t}C_2 \\ x^{2} &= e^{b_1 t}b_1 C_1 + e^{b_2 t}b_2 C_2 \end{aligned} \tag{8.5}$$

According to initial condition, the system of equations

$$\begin{gathered} C_1 + C_2 = 0 \\ b_1 C_1 + b_2 C_2 = 1 \end{gathered} \tag{8.6}$$

follows from the equality (8.5). The equality

$$\begin{gathered} C_2 = -C_1 \\ (b_1 - b_2)C_1 = 1 \end{gathered} \tag{8.7}$$

follows from the system of equations (8.6).

Example 8.2. *Let* $b_1 = i$, $b_2 = j$. *The equality*

$$(i - j)C_1 = 1 \tag{8.8}$$

follows from the system of equations (8.7). *The equality*

$$C_1 = \frac{1}{2}(-i + j) \tag{8.9}$$

follows from the equality (8.8). *Our goal is to verify whether map*

$$\begin{aligned} x^{1} &= (e^{it} - e^{jt})\frac{j - i}{2} \\ x^{2} &= (ie^{it} - je^{jt})\frac{j - i}{2} \end{aligned} \tag{8.10}$$

is solution of system of differential equations (8.1). *The equality*

$$\begin{aligned} \frac{dx^{1}}{dt} &= (ie^{it} - je^{jt})\frac{j - i}{2} = x^{2} \\ \frac{dx^{2}}{dt} &= (i^2 e^{it} - j^2 e^{jt})\frac{j - i}{2} = -x^{1} \end{aligned} \tag{8.11}$$

follows from the equality (8.10). *Therefore, the map* (8.10) *is solution of system of differential equations* (8.1) *which satisfies to initial condition*

$$t = 0 \quad x^{1} = 0 \quad x^{2} = 1$$

□

Theorem 8.3. *We can represent general solution of the system of differential equations*

$$\begin{aligned} \frac{dx^{1}}{dt} &= x^{2} \\ \frac{dx^{2}}{dt} &= -x^{1} \end{aligned} \tag{8.12}$$

as

$$
\begin{aligned}
x^{1} &= \sin t\ c_1^{1} + \cos t\ c_2^{1} \\
x^{2} &= \cos t\ c_1^{2} - \sin t\ c_2^{2} \\
c_1^{1} &= c_1^{10} + c_1^{11} i + c_1^{12} j + c_1^{13} k \\
c_1^{2} &= c_1^{20} + c_1^{21} i + c_1^{22} j + c_1^{23} k \\
c_2^{1} &= c_2^{10} + c_2^{11} i + c_2^{12} j + c_2^{13} k \\
c_2^{2} &= c_2^{20} + c_2^{21} i + c_2^{22} j + c_2^{23} k
\end{aligned}
\tag{8.13}
$$

Proof. Let

$$
\begin{aligned}
x^{1} &= x^{10} + x^{11} i + x^{12} j + x^{13} k \\
x^{2} &= x^{20} + x^{21} i + x^{22} j + x^{23} k
\end{aligned}
\tag{8.14}
$$

be representation of maps x^{1}, x^{2} relative to the basis $\overline{\overline{e}} = (1, i, j, k)$. Then

$$
\begin{aligned}
\frac{dx^{1}}{dt} &= \frac{dx^{10}}{dt} + \frac{dx^{11}}{dt} i + \frac{dx^{12}}{dt} j + \frac{dx^{13}}{dt} k \\
\frac{dx^{2}}{dt} &= \frac{dx^{20}}{dt} + \frac{dx^{21}}{dt} i + \frac{dx^{22}}{dt} j + \frac{dx^{23}}{dt} k
\end{aligned}
\tag{8.15}
$$

and we can write system of differential equations (8.12) as 4 independent systems of differential equations in real field

$$
\begin{aligned}
\frac{dx^{1i}}{dt} &= x^{2i} \\
\frac{dx^{2i}}{dt} &= -x^{1i} \\
i &= 0, 1, 2, 3
\end{aligned}
\tag{8.16}
$$

We can write solution of the systems of differential equations (8.16) as

$$
\begin{aligned}
x^{1i} &= \sin t\ c_1^{1i} + \cos t\ c_2^{1i} \\
x^{2i} &= \cos t\ c_1^{2i} - \sin t\ c_2^{2i} \\
i &= 0, 1, 2, 3
\end{aligned}
\tag{8.17}
$$

The equality (8.13) follows from the equality (8.17). □

9. Wronskian Matrix

We consider a matrix

$$
X[x_1, ..., x_m](t) = \begin{pmatrix} x_1^{1} & ... & x_m^{1} \\ ... & ... & ... \\ x_1^{n} & ... & x_m^{n} \end{pmatrix}
$$

whose columns

$$
x_1 = \begin{pmatrix} x_1^{1} \\ ... \\ x_1^{n} \end{pmatrix} \quad ... \quad x_m = \begin{pmatrix} x_m^{1} \\ ... \\ x_m^{n} \end{pmatrix}
$$

are solutions of the system of differential equations

$$\begin{aligned}\frac{dx^1}{dt}&=a^1_1x^1+...+a^1_nx^n\\ &......\\ \frac{dx^n}{dt}&=a^n_1x^1+...+a^n_nx^n\end{aligned}\tag{9.1}$$

to answer the question whether columns of the matrix are right linearly dependent. Such matrix is called **Wronskian matrix**.

Now we are ready to return to analysis of the system of differential equations

$$\frac{dx^1}{dt}=x^2$$
$$\frac{dx^2}{dt}=-x^1$$

in quaternion algebra. Consider solutions

$$x_i(t)=e^{it}\begin{pmatrix}1\\ i\end{pmatrix}\qquad x_j(t)=e^{jt}\begin{pmatrix}1\\ j\end{pmatrix}\qquad x_k(t)=e^{kt}\begin{pmatrix}1\\ k\end{pmatrix}$$

$$x(t)=\begin{pmatrix}e^{it}\frac{j-i}{2}-e^{jt}\frac{j-i}{2}\\ e^{it}i\frac{j-i}{2}-e^{jt}j\frac{j-i}{2}\end{pmatrix}$$

It is easy to see that columns of matrix

$$X[x_i,x_j,x](t)=\begin{pmatrix}e^{it}&e^{jt}&e^{it}\frac{j-i}{2}-e^{jt}\frac{j-i}{2}\\ e^{it}i&e^{jt}j&e^{it}i\frac{j-i}{2}-e^{jt}j\frac{j-i}{2}\end{pmatrix}$$

are linearly dependent from right and coordinates of column $x(t)$ with respect to columns $x_i(t)$, $x_j(t)$ do not depend on t

$$x(t)=x_i(t)\frac{j-i}{2}-x_j(t)\frac{j-i}{2}$$

Theorem 9.1. *Column vectors $x_i(t)$, $x_j(t)$, $x_k(t)$ are right linearly dependent in quaternion algebra.*

Proof. Since column vectors $x_i(t)$, $x_j(t)$, $x_k(t)$ are symetric in Wronskian matrix $X[x_i,x_j,x_k](t)$, then it does not matter to us which vectors we choose as basis. For instance, we consider linear dependence of column $x_k(t)$ with respect to columns $x_i(t)$, $x_j(t)$.

To find coefficients c_i, c_j of expansion of column $x_k(t)$ with respect to columns $x_i(t)$, $x_j(t)$

$$x_k(t)=x_i(t)c_i+x_j(t)c_j$$

we need to solve the system of linear equations

$$e^{it}c_i+e^{jt}c_j=e^{kt}\tag{9.2}$$

$$e^{it}ic_i+e^{jt}jc_j=e^{kt}k\tag{9.3}$$

According to the theorem 3.2, ${}_*{}^*$-quasideterminant of the matrix

$$a=\begin{pmatrix}e^{it}&e^{jt}\\e^{it}i&e^{jt}j\end{pmatrix}$$

of the system linear equations (9.2), (9.3) has the following form

$$\det({}_*{}^*)a=\begin{pmatrix}(1-k)e^{it}&(1+k)e^{jt}\\(i-j)e^{it}&(j-i)e^{jt}\end{pmatrix}$$

and ${}_*{}^*$-inverse matrix has the following form

$$a^{-1_*{}^*}=\frac{1}{2}\begin{pmatrix}e^{-it}(1+k)&e^{-it}(j-i)\\e^{-jt}(1-k)&e^{-jt}(i-j)\end{pmatrix}$$

Therefore,

$$\begin{pmatrix}x^1\\x^2\end{pmatrix}=\frac{1}{2}\begin{pmatrix}e^{-it}(1+k)&e^{-it}(j-i)\\e^{-jt}(1-k)&e^{-jt}(i-j)\end{pmatrix}{}_*{}^*\begin{pmatrix}e^{kt}\\ke^{kt}\end{pmatrix}\tag{9.4}$$

The equality

$$\begin{pmatrix}x^1\\x^2\end{pmatrix}=\frac{1}{2}\begin{pmatrix}e^{-it}(1+k+i+j)e^{kt}\\e^{-jt}(1-k-j-i)e^{kt}\end{pmatrix}\tag{9.5}$$

follows from the equality (9.4). The equalities

$$x^1=1+i+j+k\tag{9.6}$$

$$x^2=1-i-j-k\tag{9.7}$$

follows from the equality (9.5) and equalities

$$e^{it}=\cos t+i\sin t$$
$$e^{jt}=\cos t+j\sin t$$
$$e^{kt}=\cos t+k\sin t$$

The theorem follows from equalities (9.6), (9.7). □

Theorem 9.2.

$$e^{kt}=e^{it}\frac{1+i+j+k}{2}+e^{jt}\frac{1-i-j-k}{2}\tag{9.8}$$

PROOF. The theorem follows from equalities (9.2), (9.6), (9.7). □

10. Homogeneous Differential Equation with Constant Coefficients

10.1. Coefficients are Written from Left. The differential equation

$$\frac{d^n y}{dt^n}+a_n\frac{d^{n-1}y}{dt^{n-1}}+...+a_1 y=0 \tag{10.1}$$

is called homogeneous differential equation with constant coefficients. Using the set of variables

$$x^1=y\quad x^2=\frac{dy}{dt}\quad ...\quad x^n=\frac{d^{n-1}y}{dt^{n-1}} \tag{10.2}$$

we can write the differential equation (10.1) as the system of differential equations

$$\begin{aligned}&\frac{dx^1}{dt}=x^2\quad \frac{dx^2}{dt}=x^3\quad ...\quad \frac{dx^{n-1}}{dt}=x^n\\ &\frac{dx^n}{dt}=-a_1x^1-...-a_nx^n\end{aligned} \tag{10.3}$$

We represent the set of variables (10.2) as A^R-column

$$x=\begin{pmatrix}x^1\\ x^2\\ ...\\ x^n\end{pmatrix}$$

Then the system of differential equations (10.3) gets the form

$$\begin{pmatrix}\dfrac{dx^1}{dt}\\ \dfrac{dx^2}{dt}\\ ...\\ \dfrac{dx^{n-1}}{dt}\\ \dfrac{dx^n}{dt}\end{pmatrix}=\begin{pmatrix}0&1&0&...&0\\ 0&0&1&...&0\\ ...&...&...&...&...\\ 0&0&0&...&1\\ -a_1&-a_2&-a_3&...&-a_n\end{pmatrix}{}_*{}^*\begin{pmatrix}x^1\\ x^2\\ ...\\ x^{n-1}\\ x^n\end{pmatrix}$$

Theorem 10.1. *The solution of the differential equation* (10.1) *has form* $y=e^{bt}c$ *where* b *is* ${}_*{}^*$*-eigenvalue of the matrix*

$$\begin{pmatrix}0&1&0&...&0\\ 0&0&1&...&0\\ ...&...&...&...&...\\ 0&0&0&...&1\\ -a_1&-a_2&-a_3&...&-a_n\end{pmatrix}$$

and

$$b\in\bigcap_{i=1}^{n}Z(A,a_i)$$

or $c \in Z(A, b)$.

PROOF. The theorem follows from theorems 6.1, 6.2. $\square$

Theorem 10.2. *The solution of the differential equation* (10.1) *has form* $y = e^{bt}c$ *where* b *is root of the polynomial*

$$b^n + a_n b^{n-1} + ... + a_1 = 0 \tag{10.4}$$

PROOF. According to the theorem 5.8, the equality

$$b^n e^{bt} c + a_n b^{n-1} e^{bt} c + ... + a_1 e^{bt} c = 0 \tag{10.5}$$

follows from the equality (10.1). Since, in general, $e^{bt}c \neq 0$, then the equality (10.4) follows from the equality (10.5). $\square$

10.2. **Coefficients are Written from Right.** The differential equation

$$\frac{d^n y}{dt^n} + \frac{d^{n-1} y}{dt^{n-1}} a_n + ... + y a_1 = 0 \tag{10.6}$$

is called homogeneous differential equation with constant coefficients. Using the set of variables

$$x^1 = y \qquad x^2 = \frac{dy}{dt} \qquad ... \qquad x^n = \frac{d^{n-1} y}{dt^{n-1}} \tag{10.7}$$

we can write the differential equation (10.6) as the system of differential equations

$$\begin{array}{l} \dfrac{dx^1}{dt} = x^2 \qquad \dfrac{dx^2}{dt} = x^3 \qquad ... \qquad \dfrac{dx^{n-1}}{dt} = x^n \\ \dfrac{dx^n}{dt} = -x^1 a_1 - ... - x^n a_n \end{array} \tag{10.8}$$

We represent the set of variables (10.7) as A-column

$$x = \begin{pmatrix} x^1 \\ x^2 \\ ... \\ x^n \end{pmatrix}$$

Then the system of differential equations (10.8) gets the form

$$\begin{pmatrix} \dfrac{dx^1}{dt} \\ \dfrac{dx^2}{dt} \\ ... \\ \dfrac{dx^{n-1}}{dt} \\ \dfrac{dx^n}{dt} \end{pmatrix} = \begin{pmatrix} x^1 \\ x^2 \\ ... \\ x^{n-1} \\ x^n \end{pmatrix} {}^*{}_* \begin{pmatrix} 0 & 1 & 0 & ... & 0 \\ 0 & 0 & 1 & ... & 0 \\ ... & ... & ... & ... & ... \\ 0 & 0 & 0 & ... & 1 \\ -a_1 & -a_2 & -a_3 & ... & -a_n \end{pmatrix}$$

Theorem 10.3. *The solution of the differential equation* (10.6) *has form* $y = ce^{bt}$ *where* b *is* ${}^*{}_*$*-eigenvalue of the matrix*

$$\begin{pmatrix} 0 & 1 & 0 & ... & 0 \\ 0 & 0 & 1 & ... & 0 \\ ... & ... & ... & ... & ... \\ 0 & 0 & 0 & ... & 1 \\ -a_1 & -a_2 & -a_3 & ... & -a_n \end{pmatrix}$$

and

$$b \in \bigcap_{i=1}^{n} Z(A, a_i)$$

or $c \in Z(A, b)$.

Theorem 10.4. *The solution of the differential equation* (10.6) *has form* $y = ce^{bt}$ *where* b *is root of the polynomial*

$$b^n + b^{n-1}a_n + ... + a_1 = 0 \tag{10.9}$$

Proof. According to the theorem 5.8, the equality

$$ce^{bt}b^n + ce^{bt}b^{n-1}a_n + ... + ce^{bt}a_1 = 0 \tag{10.10}$$

follows from the equality (10.6). Since, in general, $ce^{bt} \neq 0$, then the equality (10.9) follows from the equality (10.10). □

11. Eigenvalue of multiplicity 2

Consider system of differential equations

$$\begin{aligned} \frac{dx^1}{dt} &= jx^1 \\ \frac{dx^2}{dt} &= x^1 + jx^2 \end{aligned} \tag{11.1}$$

over quaternion algebra. ${}_*{}^*$-eigenvalues of the matrix

$$a = \begin{pmatrix} j & 0 \\ 1 & j \end{pmatrix}$$

satisfy to request that the matrix

$$a - bE = \begin{pmatrix} j-b & 0 \\ 1 & j-b \end{pmatrix}$$

is ${}_*{}^*$-singular matrix. To find appropriate values of b, it is enough to consider quasideterminant

$$\det({}_*{}^*)^1_2\,(a - bE) = -(j-b)^2$$

Therefore, $b = j$ is eigenvalue of multiplicity 2.

Same as in commutative algebra, we consider fundamental solutions

$$x_1 = \begin{pmatrix} x_1^1 \\ x_1^2 \end{pmatrix} = e^{jt}\begin{pmatrix} c_1^1 \\ c_1^2 \end{pmatrix} \qquad x_2 = \begin{pmatrix} x_2^1 \\ x_2^2 \end{pmatrix} = te^{jt}\begin{pmatrix} c_2^1 \\ c_2^2 \end{pmatrix}$$

where columns c_1, c_2 satisfy condition of theorems 6.1, 6.2.

Question 11.1. *When a set of ${}_*{}^*$-eigenvalues is finite, it is easy to see multiple ${}_*{}^*$-eigenvalues. How can we find multiple ${}_*{}^*$-eigenvalues when a set of ${}_*{}^*$-eigenvalues is infinite?* □

12. Covariance

Let V be right D-vector space. Let $\overline{\overline{e}}$ be the basis with respect to which we wrote down the system of differential equations

$$\begin{aligned}\frac{dx^1}{dt} &= a^1_1 x^1 + ... + a^1_n x^n \\ &...... \\ \frac{dx^n}{dt} &= a^n_1 x^1 + ... + a^n_n x^n\end{aligned}\tag{12.1}$$

Then we can write the system of differential equations (12.1) in covariant form

$$e_i \frac{dx^i}{dt} = e_i a^i_j x^j \tag{12.2}$$

Vectors

$$\frac{dx}{dt} = e_i \frac{dx^i}{dt} \qquad x = e_i x^i$$

do not depend on the choice of basisi $\overline{\overline{e}}$. Let the basis $\overline{\overline{e}}$ map into the basis $\overline{\overline{e}}_1$

$$e_{1i} = e_j b^j_i \tag{12.3}$$

The rule of transformation of coordinates of vector x

$$e_{1i} x^i_1 = e_j b^j_i x^i_1 = e_j x^j \tag{12.4}$$

follows from the equality (12.3). The equality

$$b^j_i x^i_1 = x^j \tag{12.5}$$

follows from the equality (12.4). If we differentiate the equality (12.5), then we get

$$b^j_i \frac{dx^i_1}{dt} = \frac{db^j_i x^i_1}{dt} = \frac{dx^j}{dt} \tag{12.6}$$

From the equality (12.6), it follows the good news that the vector $\dfrac{dx}{dt}$ does not change when transforming the basis. Therefore, the system of differential equations (9.1) gets form

$$\frac{dx^i_1}{dt} = a_1{}^i_j x^j_1 \tag{12.7}$$

with respect to the basis $\overline{\overline{e}}_1$.

The equality

$$x^i_1 = b^{-1}{}^i_j x^j \tag{12.8}$$

follows from the equality (12.5). The equality

$$\frac{dx^j}{dt} = b^j_i \frac{dx^i_1}{dt} = b^j_i a_1{}^i_l x^l_1 = b^j_i a_1{}^i_l b^{-1}{}^l_k x^k = a^j_k x^k \tag{12.9}$$

follows from equalities (9.1), (12.6), (12.7), (12.8). The equality

$$b^j_i a_1{}^i_l b^{-1}{}^l_k = a^i_k \tag{12.10}$$

follows from the equality (12.9).

In commutative D-algebra, the transformation (12.10) preserves eigenvalues, because determinant of product of matrices equals to product of determinants. In noncommutative D-algebra, ${}_*{}^*$-eigenvalues may change.

13. Helpful Theorems and Proofs

This paper is a fragment of the paper
[3] Aleks Kleyn, Differential Equation over Banach Algebra,
eprint arXiv:1801.01628 (2018)
Now I am preparing new version of the paper and I expect to submit it to arXiv in August, September this year.

Theorem 13.1.

$$(a_*{}^*b)^T = a^T{}^*{}_*b^T \tag{13.1}$$

Definition 13.2. *Let V be left A-vector spaces .Let $v = (v_i \in V, i \in I)$ be set of vectors. The expression $w^i v_i$ is called* **linear combination** *of vectors v_i. A vector $\overline{w} = w^i v_i$ is called* **linearly dependent** *on vectors v_i.* □

Theorem 13.3. *Let A be associative division D-algebra. The set of vectors $\overline{\overline{e}} = (e_i, i \in I)$ is a* **basis of left A-vector space** *V if vectors e_i are linearly independent and any vector $v \in V$ linearly depends on vectors e_i.*

Theorem 13.4. *Coordinates of vector $v \in V$ relative to basis $\overline{\overline{e}}$ of left A-vector space V are uniquely defined.*

Let a_T^S be the minor matrix obtained from the matrix a by selecting rows with an index from the set S and columns with an index from the set T. Let $k = |S| = |T|$.

Definition 13.5. *If minor matrix a_T^S is ${}_*{}^*$-nonsingular matrix then we say that ${}_*{}^*$-rank of matrix a is not less then k.* **${}_*{}^*$-rank of matrix** *a, $\operatorname{rank}_{{}_*{}^*} a$, is the maximal value of k. We call an appropriate minor matrix the* **${}_*{}^*$-major minor matrix**. □

Definition 13.6. *If minor matrix a_T^S is ${}^*{}_*$-nonsingular matrix then we say that ${}^*{}_*$-rank of matrix a is not less then k.* **${}^*{}_*$-rank of matrix** *a, $\operatorname{rank}_{{}^*{}_*} a$, is the maximal value of k. We call an appropriate minor matrix the* **${}^*{}_*$-major minor matrix**. □

Theorem 13.7. *Let matrix a have n columns. If*

$$\operatorname{rank}_{{}_*{}^*} a = k < n$$

then columns of the matrix are right linearly dependent

$$a_*{}^*\lambda = 0$$

Theorem 13.8. *Let matrix a have n columns. If*

$$\operatorname{rank}_{{}^*{}_*} a = k < n$$

then columns of the matrix are left linearly dependent

$$\lambda^*{}_*a = 0$$

Theorem 13.9. *Let A-number b be ${}_*{}^*$-eigenvalue of the matrix a. The set of ${}_*{}^*$-eigencolumns of matrix a corresponding to ${}_*{}^*$-eigenvalue b is right A-vector space of columns.*

Theorem 13.10. *Let A-number b be ${}^*{}_*$-eigenvalue of the matrix a. The set of ${}^*{}_*$-eigencolumns of matrix a corresponding to ${}^*{}_*$-eigenvalue b is left A-vector space of columns.*

Proof of the Theorem 5.5:

Proof. Let the statemant (5.4) be true. According to the theorem 5.4, to prove the equality (5.5), it is enough to prove the equality

$$a^n c = ca^n \tag{13.2}$$

We will prove the equality (13.2) by induction over n.

For $n = 0$, the equality (13.2) is evident since $a^0 = 1$. According to the theorem 5.1, for $n = 1$, the equality (13.2) follows from the equality

$$ca = ac \tag{13.3}$$

Let the equality (13.2) be true for $n = k$

$$a^k c = ca^k \tag{13.4}$$

The equality

$$a^{k+1}c = aa^k c = aca^k = caa^k = ca^{k+1}$$

follows from equalities (13.3), (13.4). Therefore, the equality (13.2) is true for $n = k + 1$. □

Proof of the Theorem 5.11:

Proof. The equality

$$\frac{dx}{dt} = \frac{dce^{at}}{dt} = c\frac{de^{at}}{dt} = ce^{at}a = xa$$

follows from the theorem 5.8.

To the left of the exponent, we wrote an arbitrary constant on which the solution depends. To answer the question whether we can write a constant to the right of the exponent, we consider the lemma 13.11.

Lemma 13.11. *Let A be Banach D-algebra and $a \in A$. For any A-numbers c_1, c_2, the map*

$$x = c_1 e^{at} c_2 \tag{13.5}$$

is solution of the differential equation (5.14) iff $c_2 \in Z(A, a)$.

Proof. The equality

$$\frac{dx}{dt} = \frac{dc_1 e^{at} c_2}{dt} = c_1 \frac{de^{at}}{dt} c_2 = c_1 e^{at} a c_2$$

follows from the theorem 5.8. If $c_2 \notin Z(A, a)$, then the condition

$$c_2 a = ac_2 \tag{13.6}$$

is not true and the map (13.5) is not a solution of the differential equation (5.14).

⊙

According to the theorem 5.5, if $c_2 \in Z(A,a)$, then the map (13.5) gets form

$$x = c_1 e^{at} c_2 = c_1 c_2 e^{at} \tag{13.7}$$

and is the map of the form (5.13).

Therefore, the set of solutions (5.13) is left A-vector space. □

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Special Symbols and Notations