Brunnian link maps in the 4-sphere

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A *link map* is a map $X_1 \sqcup \cdots \sqcup X_m \to Y$ such that the images of the X_i are pairwise disjoint, and a *link homotopy* is a homotopy whose every time instant is a link map. For example, link maps $S^p \sqcup S^q \to S^{p+q+1}$ are classified (up to link homotopy) by the linking number, and link maps $S^1 \sqcup S^1 \sqcup S^1 \to S^3$ are classified by Milnor's triple $\bar{\mu}$-invariant. A nontrivial link map $S^2 \sqcup S^2 \to S^4$ was constructed by R. Fenn and D. Rolfsen (1986) using that each component of the Whitehead link is null-homotopic in the complement ot the other one. P. Kirk (1988) introduced an invariant of link maps $S^2 \sqcup S^2 \to S^4$ with values in the infinitely generated free abelian group $\mathbb{Z}[x] \oplus \mathbb{Z}[y]$ and found its image. According to a 2017 preprint by R. Schneiderman and P. Teichner, the long-standing problem of injectivity of Kirk's invariant has an affirmative solution.

A natural extension of Kirk's invariant to link maps of m copies of S^2 in S^4 was described by U. Koschorke (1991) and takes values in $\big(\mathbb{Z}[\mathbb{Z}^{m-1}/t]\big)^m$, where $t(\vec{v}) = -\vec{v}$. We show that the Kirk–Koschorke invariant is not injective for $m > 2$. To prove this, we introduce a new "non-abelian" invariant of m-component link maps in S^4 with values in $\big(\mathbb{Z}[RF_{m-1}/T, c]\big)^m$, where RF_k is the Milnor free group (RF_2 is also known as the discrete Heisenberg group), $T(g) = g^{-1}$ and c stands for conjugation. Loosely speaking, the new invariant is related to the Kirk–Koschorke invariant in the same way as Milnor's $\bar{\mu}$-invariants of link homotopy are related to pairwise linking numbers.

For link maps $S^2 \sqcup S^2 \sqcup S^2 \to S^4$ we also find the image of their Kirk–Koschorke invariant. The main step is a new elementary construction of Brunnian link maps $S^2 \sqcup \cdots \sqcup S^2 \to S^4$: they are described by an explicit link-homotopy-movie (just like the Fenn–Rolfsen link map), which is closely related to the minimal solution of the Chinese Rings puzzle. The existence of nontrivial Brunninan link maps in S^4 (of more than two components) was established previously by Gui-Song Li (1999), but his construction requires a lot more of 4-dimensional imagination (it is based on iterated Whitney towers and a process of their desingularization) and does not suffice to generate the image of the Kirk–Koschorke invariant.

Our interest in finding the images of invariants of link maps in S^4 actually comes from a study of classical links. In arXiv:1711.03514 = JKTR 27:13 (2018), 1842012, the computation of the image of Kirk's invariant of link maps $S^2 \sqcup S^2 \to S^4$ is applied to reprove the Nakanishi–Ohyama classification of links $S^1 \sqcup S^1 \to S^3$ up to self C_2-moves. Now, our computation of the image of the Kirk–Koschorke invariant of link maps $S^2 \sqcup S^2 \sqcup S^2 \to S^4$ has the following application: Two links $S^1 \sqcup S^1 \sqcup S^1 \to S^3$ that are link homotopic to the unlink are related by C_2^{xxx}-moves (=self C_2-moves) and $C_3^{xx,yz}$-moves (of Goussarov and Habiro) if and only if they have equal $\bar{\mu}$-invariants (with possibly repeating indices) of length at most 4.