Optimal Growth – Optimal Control – Ramsey Saving

Dynamics of Optimal Intertemporal Consumption and Saving :

The Solutions of Persistent Growth Models with Normative Capital Accumulation

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Introduction

Optimal saving-consumption study by Ramsey (1928)

One of the most important **decision** (***control***) **variables** in models for individuals or nations is the choice about the **normative** (*optimal*) **sizes** of the **saving rates**, i.e. about the **share of income** to be devoted to **investment** (***capital accumulation***) and hence **not available** for ***consumption***.

Newbery (2008) has stated: "**Ramseys formulation** of the problem **served** as a **model** for almost all subsequent studies of **optimal economic growth**, and, with the ***critical addition*** of a ***growing population***, might have created ***neoclassical growth theory*** about 30 years before **Solows** (1956) contribution."

Keynes (1933) wrote: "It is, I think, one of the most remarkable contributions to ***mathematical economics*** ever made."

Introduction

Optimal saving-consumption study by Cass (1965)

Cass (1965) mathematically demonstrated the ***convergence*** of an ***initial capital-labor ratio*** to unique positive ***steady state ratio*** ('optimal balanced growth path') - ***replacing*** the stability issues of ***stationary state*** of Ramsey (1928).

The Cass ***optimal growth (control)*** model is now one of the most important theoretical ***paradigms*** for ***dynamic macroeconomic*** models.

Introduction

Qualitative and Quantitative Properties of Dynamic Models

We need to study ***growth models*** from a ***quantitative standpoint***, before one can ***claim*** to have ***explained*** and ***parametrically*** accounted for ***major differences*** in economic ***growth over time*** and ***across counties*** or other empirical ***policy analysis*** and sound ***advice***.

Besides ***initial conditions***, ***solutions (time paths)*** of ***optimal growth*** models (***control*** problems, ***systems of differential equations***) depend critically on several important ***parameters*** involved in the basic ***technology (production function)*** and ***preference (utility function)*** ***assumptions***.

Introduction

Challenge - de La Grandville (2018) – MD Macro Dynamics

"***Optimal growth theory***, as it stands today, does ***not work***.

Using ***strictly concave utility functions*** systematically ***inflicts*** on the economy ***distortions*** that are either historically ***unobserved*** or ***unacceptable*** by society."

Response to de La Grandville

Economists need a ***clear*** and ***better understanding*** of the ***Euler*** and

Pontryagin dynamic economic equations,

both from an ***analytical economic*** and ***computational*** point of view.

Introduction – Purpose

The ***purpose*** is to derive and solve, rigorously and ***quantitatively***, the ***dynamics*** of the ***optimal growth (saving)*** model for ***general production*** and ***utility functions*** - ***f(k), u(c)***.

Our main ***theoretical*** results are ***Theorem 1*** and ***Lemma 1-3***.

Quantitative Applications

T***ime paths*** are actually demonstrated by ***parametric*** benchmark ***quantitative solutions*** of the ***optimal control*** dynamic systems.

N***umerical demonstrations*** with ***CIES*** preferences and ***CES*** technologies.

Ramsey saving model is capable of generating ***persistent endogenous growth***.

Abstract

This paper explores the *optimal saving solutions* for optimal economic growth generated endogenously by a *Ramsey model*. *Sufficient conditions* are presented for *persistent* economic growth within a standard Ramsey model.

In ***phase diagrams*** of trajectories (solutions), the ***optimal path (trajectory)*** is a ***separator***. *Below* the ***separator***, *over-saving* diminishes consumption, ***ultimately*** leading to a sub-optimal situation where *all incomes* are *saved*. *Above* the ***separator*** *under-saving* suddenly *collapses* the economy as its productive ***capital*** vanishes *to zero*.

The paper gives comprehensive *numerical applications* for *CIES preferences* and *CES technologies*, together with *parametric sensitivity* analyses of the *optimal solutions*.

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Normative Capital Accumulation - Optimal Saving

- The **representative consumer** has a time additive intertemporal *cardinal utility* function, $u\,[\,c(t)\,]\,e^{-\rho t} \;\equiv\; I(t)$, (Integrand), in **continuos time** summed for, $\forall t \in \Re$:

$$U = U[\,\widetilde{c(t)}] = \int_0^{\infty} u\,[\,c(t)\,]\,e^{-\rho t}dt, \quad \rho > 0 \;; \quad U \;=\; U(\infty) \;\equiv\; \int_0^{\infty} I(t)\,dt$$

$$U^* = U[c^*(t)] = \max\; U[\,\widetilde{c(t)}] = \max_{c(t)} \int_0^{\infty} u\,[c(t)]\,e^{-\rho t}dt, \quad \rho > 0$$

$$V = V[\widetilde{c(t)}] = \int_0^{\infty} u[c(t)]L(t)\,e^{-\rho t}dt = L_0 \int_0^{\infty} u[c(t)]\,e^{-(\rho-n)t}dt\;;\;\rho - n > 0$$

$$u'(c) > 0\,, \quad u''(c) < 0\,, \qquad \lim_{c\to 0} u'(c) = \infty, \qquad \lim_{c\to\infty} u'(c) = 0.$$

- **Dynamics** of the **capital-labor ratio** :

$$dk/dt \;=\; \dot{k} \;=\; h(k,c), \qquad k(t) \;=\; K(t)/L(t)$$

Normative Capital Accumulation – Optimal Saving

- **Production function**

$$Y = F(L,K) = Lf(k) \equiv Ly, L \neq 0; \quad F(0,0) = 0$$

- The Production function $f(k)$ is strictly *concave* and *monotonic* increasing

$$\forall k > 0: \quad f'(k) > 0, \quad f''(k) < 0; \quad k \in [0, \infty[$$

$$\lim_{k \to 0} f'(k) \equiv \bar{b} \leq \infty, \quad \lim_{k \to \infty} f'(k) \equiv \underline{b} \geq 0; \quad f'(k) \in [\underline{b}, \bar{b}]$$

- **Factor accumulation** is one-sector (good) macro models become

$$dK/dt = \dot{K} = S - \delta K = Y - C - \delta K = L[f(k) - \delta k - c]; \quad dL/dt = \dot{L} = nL$$

$$dk/dt = \dot{k} = f(k) - (n+\delta)k - c = h(k,c).$$

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The Ramsey Problem – The Hamilton Function

The **Ramsey optimization** (control, maximum) problem is:

$$\max\ V \ = \ \max\ V[\widetilde{c(t)}] \ = \ \max_{c(t)} \int_0^{\infty} u\,[\,c(t)\,]\,e^{-(\rho-n)t}dt$$

$$\text{s.t.}\quad \dot{k} \ = \ f(k) - c - (n+\delta)k \ = \ h(k,c), \quad c \geq 0,$$

is equivalent to *maximizing* the current value **Hamiltonian function**,

$$\mathcal{H}\,(c,k,\lambda) = u\,(c) + \lambda(t)\,h(k,c)\ ; \quad \mathcal{H}\,(c,k,\lambda) \ = \ u\,(c) \ + \lambda(t)\,[\,f(k) - (n+\delta)k - c\,]$$

with a **Lagrange multiplier** (**costate, adjoint**) variable, **$\lambda(t)$**, and the **transversality conditions:**

$$k(0) \ = \ k_0\,, \qquad \lim_{t\to\infty} \lambda(t)\,k(t)\,e^{-(\rho-n)t} \ = \ 0$$

First order (**necessary**) conditions by the **maximum principle** are,

$$\frac{\partial\mathcal{H}}{\partial\lambda} = \frac{dk}{dt}\ ; \qquad \frac{\partial\mathcal{H}}{\partial\lambda} \ = \ h(k,c) \ = \ f(k) - (n+\delta)k - c \ = \ \frac{dk}{dt} \equiv \dot{k}$$

$$\frac{\partial\mathcal{H}}{\partial c} = 0\ ; \qquad \frac{\partial\mathcal{H}}{\partial c} \ = \ \frac{\partial u}{\partial c} + \lambda(t)\frac{\partial h}{\partial c} \ = \ u'(c) - \lambda(t) \ = \ 0\ ; \quad \lambda(t) \ = \ u'(c)$$

and with the **necessary** (“**Euler**”) condition – a **costate** (**adjoint**) **equation** of **motion** for **λ**,

$$\frac{\partial\mathcal{H}}{\partial k} = -\frac{d\lambda}{dt} + (\rho-n)\lambda(t)\ ; \quad \frac{\partial\mathcal{H}}{\partial k} = \lambda(t)\,[\,f'(k) - (n+\delta)\,] \ = -\,\dot{\lambda}(t) \ + \ (\rho-n)\lambda(t)$$

Change of the State Variables from (λ, k) to (c, k)

- **Elimination** of λ

$$\dot{\lambda}(t) = u''(c)\,\dot{c}(t)\,; \quad \widehat{\lambda} = \frac{\dot{\lambda}}{\lambda} = \frac{du'(c)/dt}{u'(c)} = \widehat{u'(c)} = \frac{u''(c)c}{u'(c)}[\frac{\dot{c}}{c}] = E(u'(c),c)\,\widehat{c}$$

$$\dot{\lambda}(t) = -\lambda(t)\,[\,f'(k) - \delta - \rho\,]\;;\quad \widehat{\lambda} = -[\,f'(k) - \delta - \rho\,]$$

or alternatively the "Euler-Ramsey rule" as optimal changes in 'observable' *per capita consumption*, ***c***

$$\widehat{c} = \dot{c}/c = -\frac{\dot{\lambda}/\lambda}{E(u'(c),c)} = -\frac{u'(c)}{u''(c)\,c}\,[\,f'(k) - (\delta+\rho)\,]$$

and the "Euler-Ramsey rule" as one *ordinary autonomous differential* equation in (k, c) :

$$\dot{c} = c\,\eta(c)\,[\,f'(k) - (\delta+\rho)\,]\;;\quad \dot{c}/c = \eta(c)\,[\,f'(k) - (\delta+\rho)\,]$$

- **IES**, **intertemporal elasticity of substitution** : $\eta(c) = -u'(c)/[u''(c)\,c] > 0$

  or **reciprocal** : $\theta(c) = 1/\eta(c) = -E(u'(c),c) = -d\ln u'(c)/d\ln c = -E(MU(c),c) > 0.$

- We have the "**Euler-Ramsey rule**" *of Consumption, Optimal Saving,* **Capital Accumulation** :

$$[\rho - \widehat{u'(c)} = \rho - E(MU(c),c)\,\widehat{c} = \rho + \frac{\widehat{c}}{\eta(c)} = \rho - \widehat{\lambda}\,] = [\,f'(k) - \delta = MP_K(k) - \delta\,]$$

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Dynamic System of Optimal Consumption – Optimal Saving

- $\dot{k} \;=\; h(k,c) \;=\; f(k) - (n+\delta)k - c$
- $\dot{c} \;=\; g(k,c) \;=\; c\,\eta(c)\,[\,f'(k) - (\delta+\rho)\,]$
- $\lim_{t\to\infty} u'\,[\,c(t)\,]\,k(t)\,e^{-(\rho-n)t} \;=\; 0$
- **Saving rate,** $s(t) = S(t)/Y(t)$ **:**

$$s(t) \;=\; 1 \;-\; C(t)/\,Y(t) \;= 1 \;-\; c(t)/\,y(t) \;=\; 1 \;-\; c(t)/\,f(k(t))$$

- **Growth rate** of **per capita income**,

$$\widehat{y}\,(t) \;\equiv\; \dot{y}/y = E\,(f\,(k),k)\,(\dot{k}/k) = \epsilon_K(k)\,(\dot{k}/k) = \epsilon_K(k[t])\,\widehat{k}\,(t)$$

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1885–1913	1921–1939	1973–1991
$0.1 \leq s \leq 0.2$	$0.15 \leq s \leq 0.2$	$0.15 \leq s \leq 0.3$

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CES Production Function $Y = F(L, K)$

Parameters : $\gamma > 0\ ,\quad 0 < a < 1\ ,\quad \sigma > 0$

- $$Y \;=\; F(L,K) = \gamma \left[(1-a)L^{\frac{\sigma-1}{\sigma}} + aK^{\frac{\sigma-1}{\sigma}} \right]^{\frac{\sigma}{\sigma-1}}$$
- $$AP_L(k) \;=\; Y/L = y = f(k) = \gamma \left[(1-a) + ak^{\frac{\sigma-1}{\sigma}} \right]^{\frac{\sigma}{\sigma-1}}$$
- $$MP_L(k) \;=\; (1-a)\,\gamma^{\frac{\sigma-1}{\sigma}}\ [AP_L(k)]^{\frac{1}{\sigma}}$$
- $$MP_K(k) \;=\; f'(k) \;=\; a\gamma \left[a + (1-a)k^{\frac{1-\sigma}{\sigma}} \right]^{\frac{1}{\sigma-1}}\ ;\ \sigma > 1 : \underline{b} = \gamma\, a^{\frac{\sigma}{\sigma-1}}$$
- $$AP_K(k) \;=\; Y/K \;=\; f(k)\,/k\ ; \qquad MP_K(k) = a\,\gamma^{\frac{\sigma-1}{\sigma}}\ [AP_K(k)]^{\frac{1}{\sigma}}$$
- $$\epsilon_K(k) \;=\; MP_K/AP_K = \left[1 + [(1-a)/a]\,k^{\frac{1-\sigma}{\sigma}} \right]^{-1}\ ;\ \epsilon_L = 1 - \epsilon_K$$
- $$E\,(MP_L, AP_L) \;=\; 1/\sigma \;=\; E\,(MP_K, AP_K)$$

Steady State (Saddle Point) Solutions

In the **CES case**, the **dynamic system** becomes

$$\dot{k} = \gamma\left[(1-a)+ak^{\frac{\sigma-1}{\sigma}}\right]^{\frac{\sigma}{\sigma-1}} - (n+\delta)k - c$$

$$\dot{c} = c\,\eta(c)\left[\gamma a\left[a+(1-a)k^{\frac{-(\sigma-1)}{\sigma}}\right]^{\frac{1}{\sigma-1}} - \delta - \rho\right]$$

If they exist, **steady-state** values of **capital-labor ratios** and per **capita consumption** in optimal one-sector growth models are *singular/critical points*,

$$[\dot{c}=0 \Leftrightarrow k(t) = \kappa] \Leftrightarrow [MP_K(\kappa) = f'(\kappa) = \rho+\delta]$$

$$[\dot{k}=0 \Leftrightarrow c(t) = c(\kappa)] \Leftrightarrow [c(\kappa) = f(\kappa) - (n+\delta)\kappa]$$

Table 1, the optimal saving rates, *s(k)*, are always less than $\epsilon_K(\kappa)$, ("golden rule" saving rate):

$$s(\kappa) = 1 - \frac{c(\kappa)}{f(\kappa)} = \frac{(n+\delta)\,\kappa}{f(\kappa)} = \frac{n+\delta}{AP_K(\kappa)} < \epsilon_K(\kappa) = \frac{\rho+\delta}{AP_K(\kappa)}\;;\;\rho > n$$

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Parameter Values

• Real Interest Rate	MP_K	-	0.07 - 0.11	-	OECD Economies
• Real Interest Rate	MP_K	-	0.12 - 0.17	-	Poor Countries
• Depreciation Rate	δ	-	0.03 – 0.06		
• Capital intensity	a	-	0.2 – 0.6	-	Labor intensity (1-a)
• Substitution elasticity	σ	-	0.5 – 2.5		
• Total Factor Productivity	γ	-	0.3 – 3.0		
• Discount rate (Time Preference)	ρ	-	0.05 – 0.10	-	0.12 – 0.17

Table 1. **Parameters** for optimal growth models : **CES** cases with **steady states** or **asymptotics**.

Parameter values - steady state models							Model characteristics									
case	ρ	n	δ	γ	a	σ	κ	$c(\kappa)$	$f(\kappa)$	$MP_L(\kappa)$	$f(\kappa)/\kappa$	$MP_K(\kappa)$	$\epsilon_K(\kappa)$	K/Y	$s(\kappa)$	$\rho+\delta$
1	0.050	0.02	0.05	1.0	0.20	1.0	2.378	1.023	1.189	0.951	0.500	0.100	0.200	2.000	0.140	0.100
2	0.050	0.02	0.05	1.0	0.25	1.0	3.393	1.119	1.357	1.018	0.400	0.100	0.250	2.500	0.175	0.100
3	0.050	0.02	0.05	1.0	0.40	1.0	10.079	1.814	2.520	1.512	0.250	0.100	0.400	4.000	0.280	0.100
4	0.050	0.02	0.05	3.0	0.40	1.0	62.898	11.321	15.724	9.435	0.250	0.100	0.400	4.000	0.280	0.100
5	0.070	0.02	0.05	1.0	0.40	1.0	7.438	1.710	2.231	1.339	0.300	0.120	0.400	3.333	0.233	0.120
6	0.075	0.02	0.08	1.0	0.40	1.0	4.855	1.396	1.881	1.129	0.388	0.155	0.400	2.581	0.258	0.155
7	0.100	0.02	0.08	1.0	0.40	1.0	3.784	1.325	1.703	1.022	0.450	0.180	0.400	2.222	0.222	0.180
8	0.120	0.02	0.08	1.0	0.40	1.0	3.175	1.270	1.587	0.952	0.500	0.200	0.400	2.000	0.200	0.200
9	0.100	0.02	0.08	2.0	0.40	1.0	12.014	4.205	5.406	3.244	0.450	0.180	0.400	2.222	0.222	0.180
10	0.050	0.02	0.08	0.3	0.40	1.0	0.875	0.197	0.284	0.171	0.325	0.130	0.400	3.077	0.308	0.130
11	0.050	0.02	0.08	1.0	0.60	1.0	45.764	5.339	9.915	3.966	0.217	0.130	0.600	4.615	0.461	0.130
12	0.050	0.02	0.08	1.0	0.40	1.0	6.509	1.464	2.115	1.269	0.325	0.130	0.400	3.077	0.308	0.130
13	0.150	0.02	0.05	1.0	0.60	1.0	15.588	4.105	5.196	2.078	0.333	0.200	0.600	3.000	0.210	0.200
1	0.050	0.02	0.05	1.0	0.25	0.5	1.775	0.999	1.123	0.945	0.632	0.100	0.158	1.581	0.111	0.100
2	0.050	0.02	0.05	1.0	0.40	0.5	2.667	1.146	1.333	1.067	0.500	0.100	0.200	2.000	0.140	0.100
3	0.050	0.02	0.05	1.0	0.60	0.5	4.624	1.564	1.888	1.425	0.408	0.100	0.245	2.449	0.172	0.100
4	0.050	0.02	0.05	3.0	0.60	0.5	9.107	5.802	6.439	5.529	0.707	0.100	0.141	1.414	0.099	0.100
5	0.075	0.02	0.05	1.0	0.40	1.5	53.718	5.624	9.384	2.669	0.175	0.125	0.716	5.724	0.400	0.125
6	0.075	0.02	0.05	1.0	0.60	1.2	820.885	67.504	124.966	22.356	0.152	0.125	0.821	6.569	0.461	0.125
7	0.100	0.02	0.08	0.3	0.40	1.5	0.385	0.174	0.212	0.143	0.551	0.180	0.327	1.814	0.181	0.180
8	0.100	0.02	0.08	0.2	0.40	1.5	0.162	0.094	0.110	0.080	0.675	0.180	0.267	1.481	0.148	0.180
9	0.060	0.02	0.05	1.0	0.40	1.7	108201.257	4478.988	12053.076	150.937	0.111	0.110	0.987	8.977	0.631	0.110
Parameters - persistent growth models							Limits for $k \to \infty$									
1	0.100	0.02	0.05	1.0	0.60	1.5	∞	∞	∞	∞	0.216	0.216	1.000	4.630	0.630	0.150
2	0.100	0.02	0.05	1.0	0.40	2.0	∞	∞	∞	∞	0.160	0.160	1.000	6.250	0.500	0.150
3	0.060	0.02	0.08	1.0	0.40	3.0	∞	∞	∞	∞	0.253	0.253	1.000	3.952	0.842	0.140
4	0.070	0.02	0.08	1.0	0.40	7.0	∞	∞	∞	∞	0.343	0.343	1.000	2.915	0.854	0.150
5	0.080	0.02	0.05	1.0	0.40	3.0	∞	∞	∞	∞	0.253	0.253	1.000	3.952	0.763	0.130

Persistent Growth - Asymptotic Growth : Solutions - $c(t)$, $k(t)$ - and the Phase Portrait

- **Assumption 1**. *Technology*: The per capita function $f(k)$, has continuity and differentiability properties,

$$(i) \quad f(k) \in C^0([0,\infty[) \cap C^1(]0,\infty[), \qquad (ii) \quad f(0) \geq 0.$$

- Further, it is assumed for *persistent growth* that for a *concave* function with $f(k) \to \infty$ as $k \to \infty$:

$$(iii) \quad \forall k > 0: \quad f'(k) > \delta + \rho \quad \Rightarrow \quad \forall t > 0: \; \dot{c}(t) > 0\,, \quad (iv) \quad \lim_{k\to\infty} f'(k) \equiv \underline{b} > \delta + \rho.$$

No stationary solutions exist in the *closed* first quadrant $\overline{\Re}^2_+$, [except for (0, 0)]

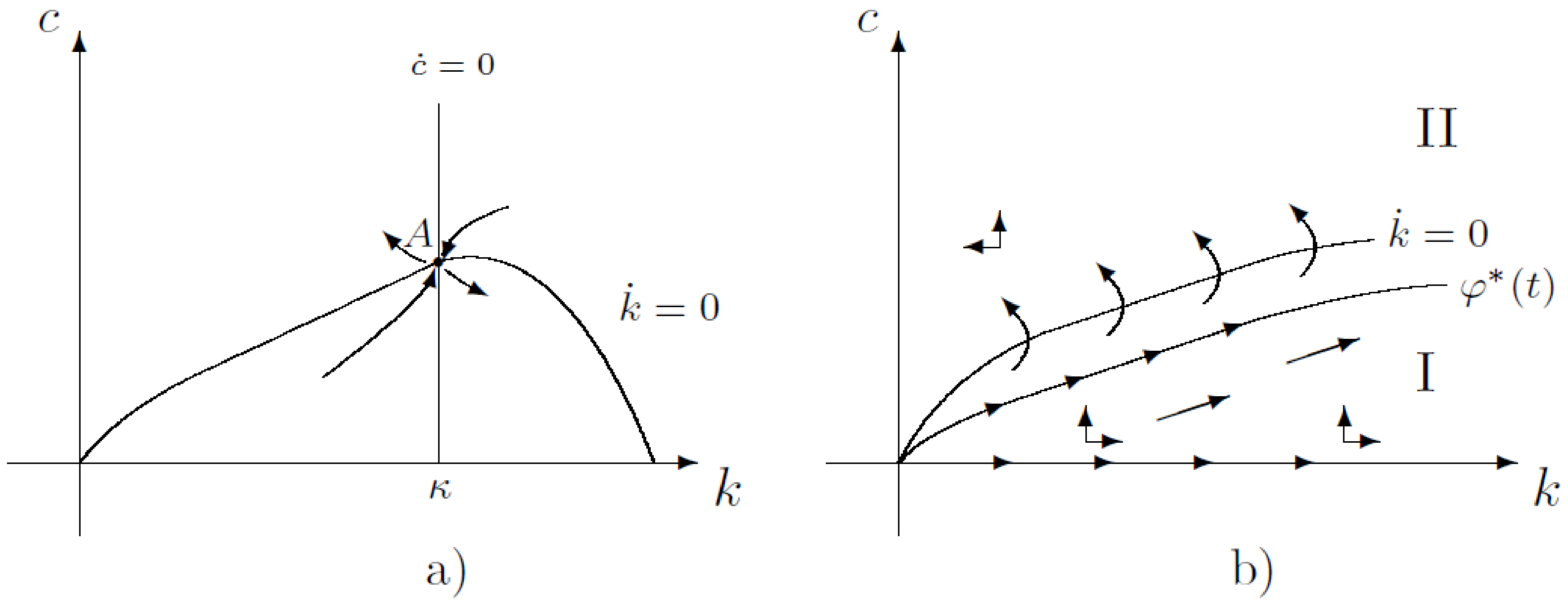

Figure 1: Dynamics with optimal saving

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Persistent Growth : Solutions – Phase Portrait

Theorem 1. Optimal (Ramsey) Saving - Persistent Endogenous per Capita Growth

$$persistent\ growth\ (\dot{c} > 0) : \lim_{k \to \infty} f'(k) \equiv \underline{b} > \rho + \delta \quad \Leftrightarrow \quad \rho < \underline{b} - \delta$$

$$\exists\, \varphi^*(t) : \sup_{c>0} \eta(c) \equiv \overline{\eta} < \frac{\underline{b} - (n + \delta)}{\underline{b} - (\rho + \delta)} \quad \Leftrightarrow \quad \rho > \frac{(\underline{b} - \delta)(\overline{\eta} - 1) + n}{\overline{\eta}}$$

$$\exists\, V : \sup_{c>0} \eta(c) \equiv \overline{\eta} < \frac{\rho - n}{(\underline{b} - \delta) - \rho} \quad \Leftrightarrow \quad \rho > \frac{(\underline{b} - \delta)\overline{\eta} + n}{1 + \overline{\eta}}$$

Separator – Optimal Trajectory $\varphi^*(t) \equiv [\, k^*(t),\ c^*(t) \,], \quad t \in \Re$ **, Figure 1**

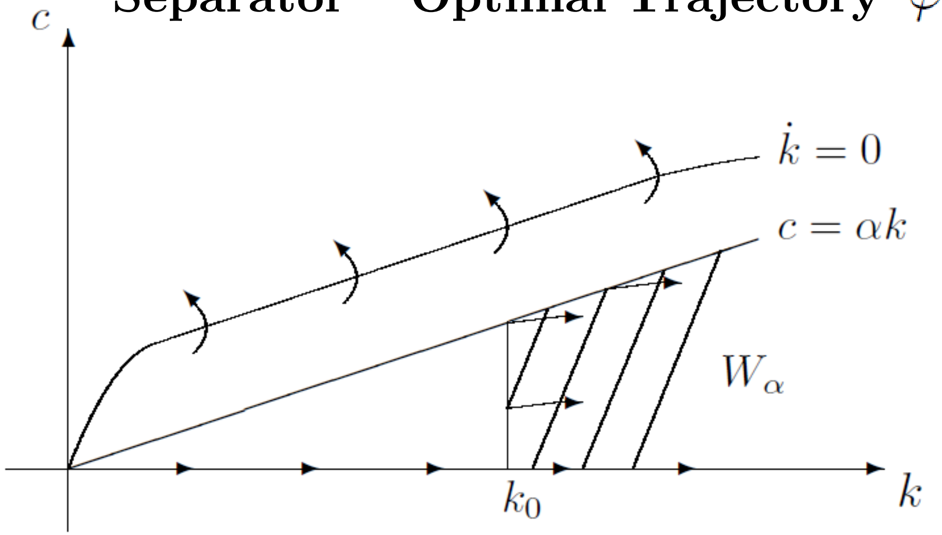

Figure 2.a: The positive invariant región $\mathbf{W}_\alpha$, with endogenous (persistent) per capita growth

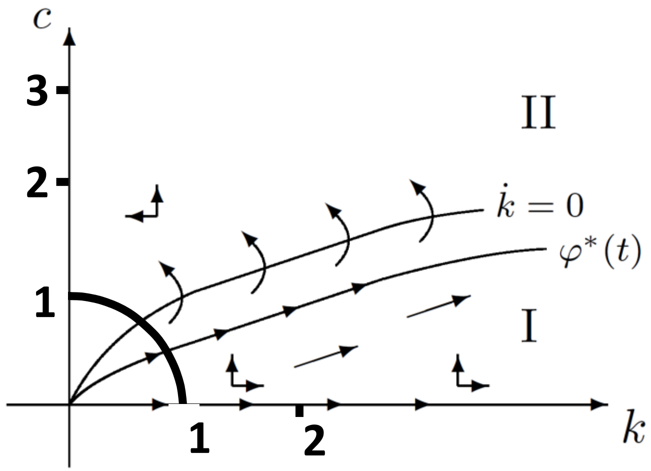

Figure 2.b: curve C $= C_I \cup C_{II} \cup \{(1,0),\ (0,1)\}$ (*closed set*), C_{II} must be an open and connected set of C. $C_I \cup \{(1,0)\}$ must be a closed set *with the end point* $[\bar{k}^*, \bar{c}^*]$ *of curve* C_I, *where* $\varphi^*(t)$ *passes.*

Coordinate Transformations and Transformed Solutions

$$z = Y/K = y/k = f(k)/k = \zeta(k)\,; \quad k = \zeta^{-1}(z)$$

$$x = C/K = c/k\,; \quad c = k\,x = \zeta^{-1}(z)\,x$$

$$\dot{z} = \phi(z,x) = z\,[\,f'(\zeta^{-1}(z))/z - 1\,]\,(\,z - x - n - \delta\,)$$

$$\dot{x} = \psi(z,x) = x\,[\eta\,(\zeta^{-1}(z)x)\,[f'(\zeta^{-1}(z)) - \delta - \rho] - z + x + n + \delta]$$

- **Saddle point** and **optimal trajectory** in **transformed coordinates** – Reversing *time* variable

$$\dot{z} = 0 : z = \underline{b} \quad ; \quad x = z - n - \delta$$

$$\dot{x} = 0 : x + \eta\,(\zeta^{-1}(z)\,x)\,\left[f'(\zeta^{-1}(z)) - \delta - \rho\right] - z + n + \delta = 0$$

- *Asymptotic growth rates* :

$$\lim_{t\to\infty} \widehat{k}(t) = \lim_{t\to\infty} \widehat{c}(t) = \lim_{t\to\infty} \widehat{y}(t) = \underline{\widehat{y}} = \overline{\eta}\,(\underline{b} - \delta - \rho)$$

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Applications – Solutions with CIES and CES

CIES : $u(c) = \frac{c^{1-\theta} - 1}{1-\theta}$, $\theta = 1 : u(c) = \ln c\,;\ -E(MU(c), c) = \frac{1}{\eta(c)} = \theta \geq 0$

Theorem 1 is *satisfied* with : $\underline{b} - \delta > \rho > (\underline{b} - \delta + n\theta)/(1+\theta)$

CIES and CES solutions with **their seven parameters :**

$$\gamma = 0.661157,\ a = 0.55,\ \sigma = 2.0,\ \delta = 0.03,\ n = 0.01,\ \theta = 1 = \eta\,,\ \rho = 0.16$$

Theorem 1 : $0.17 > \rho > 0.09$

CES case, the **dynamic system** becomes

$$\dot{k} = \gamma\left[(1-a) + ak^{\frac{(\sigma-1)}{\sigma}}\right]^{\frac{\sigma}{(\sigma-1)}} - c - (n+\delta)k$$

$$\dot{c} = (c/\theta)\left[\gamma a\left[a + (1-a)k^{\frac{-(\sigma-1)}{\sigma}}\right]^{\frac{1}{(\sigma-1)}} - \delta - \rho\right]$$

Isocline : $\dot{k} = 0 \Leftrightarrow c = \gamma\left[(1-a) + ak^{\frac{(\sigma-1)}{\sigma}}\right]^{\frac{\sigma}{(\sigma-1)}} - (n+\delta)k$

Applications – Solutions with CIES and CES

- The **transformed dynamic system** is

$$\dot{z} \;=\; \phi(z,x) \;=\; z\,[\,a\,\gamma^{\frac{\sigma-1}{\sigma}}\, z^{\frac{1-\sigma}{\sigma}} \;-1\,]\,[\,z - x - n - \delta\,]$$

$$\dot{x} \;=\; \psi(z,x) \;=\; x\,[\,x - z\,(1 - \frac{1}{\theta}\gamma^{\frac{\sigma-1}{\sigma}}\, a\, z^{\frac{1-\sigma}{\sigma}}\,) + n + \delta - \frac{\delta+\rho}{\theta}\,]$$

- The **singularities** (*saddle point* and *node* point) become,

$$z^* \;=\; \gamma\, a^{\frac{\sigma}{\sigma-1}} \;=\; \underline{b}$$

$$x^* \;=\; z^*\,(1 - 1/\theta\,) + (\,\delta + \rho\,)/\,\theta - n - \delta\,; \quad node\,:\, x^* \;=\; 0$$

- Hence the **long-run** (**asymptotic**) *saving rate* (s*) is given by,

$$s^* = 1 - x^*/\,z^* = 1 - \frac{\rho - (\,\underline{b} - \delta\,)(\,1 - \theta\,) - n\,\theta}{\underline{b}\,\theta}$$

- **CES**-baseline **parameters** give exactly the **numbers** : $\underline{b} = z^* = 0.20,\ x^* = 0.15,\ s^* = 0.25$

Transformed Solutions, Phase Portrait : CIES and CES

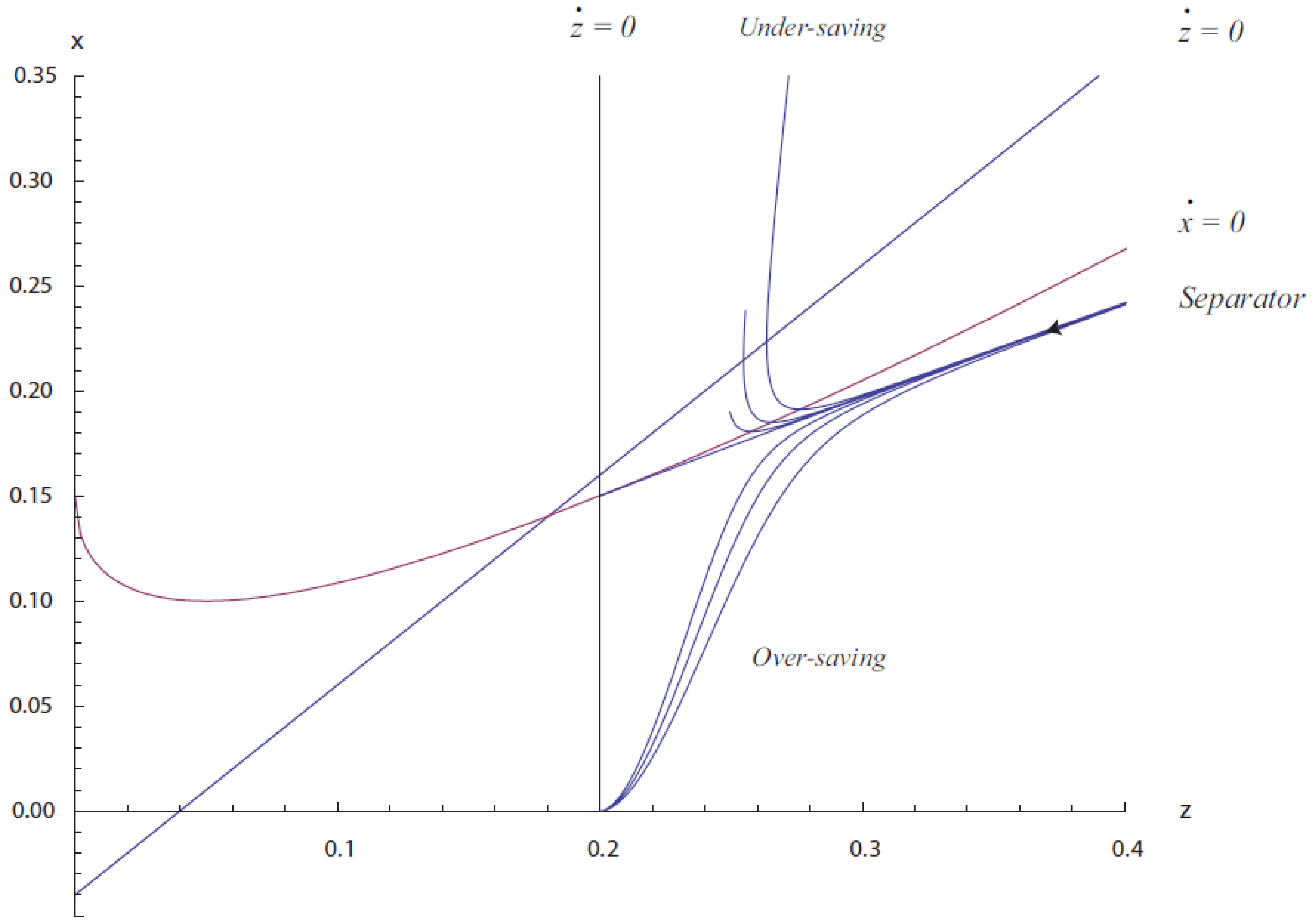

Figure 4: The *transformed* **phase portrait** in the (z, x) space. ***CES****-baseline case.*

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Solutions and Phase Portrait with CIES and CES

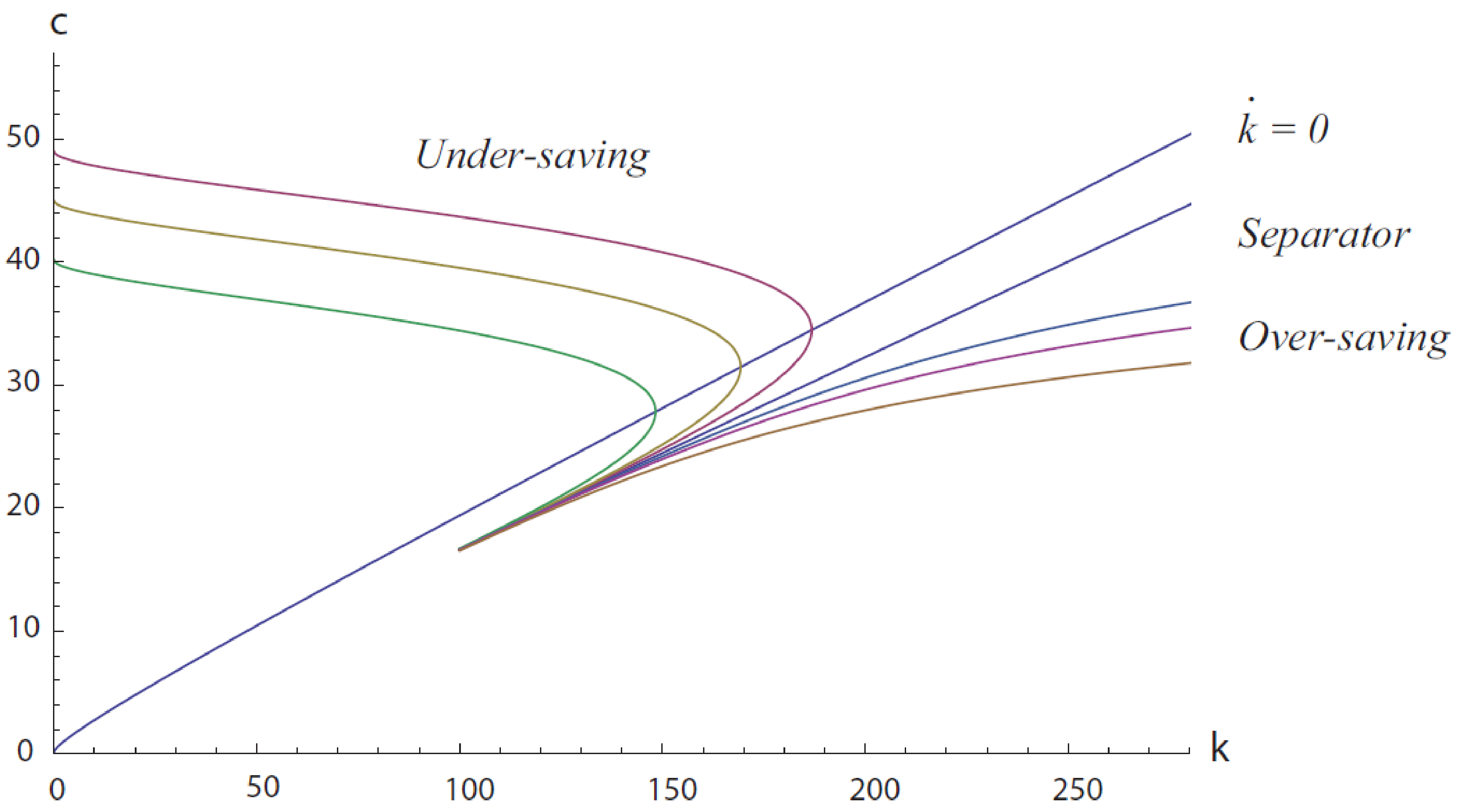

Figure 3 : The **phase portrait** in the **state** (k, c) **space**, ***CES***-*baseline case.*

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Solutions and Phase Portrait with CIES and CES

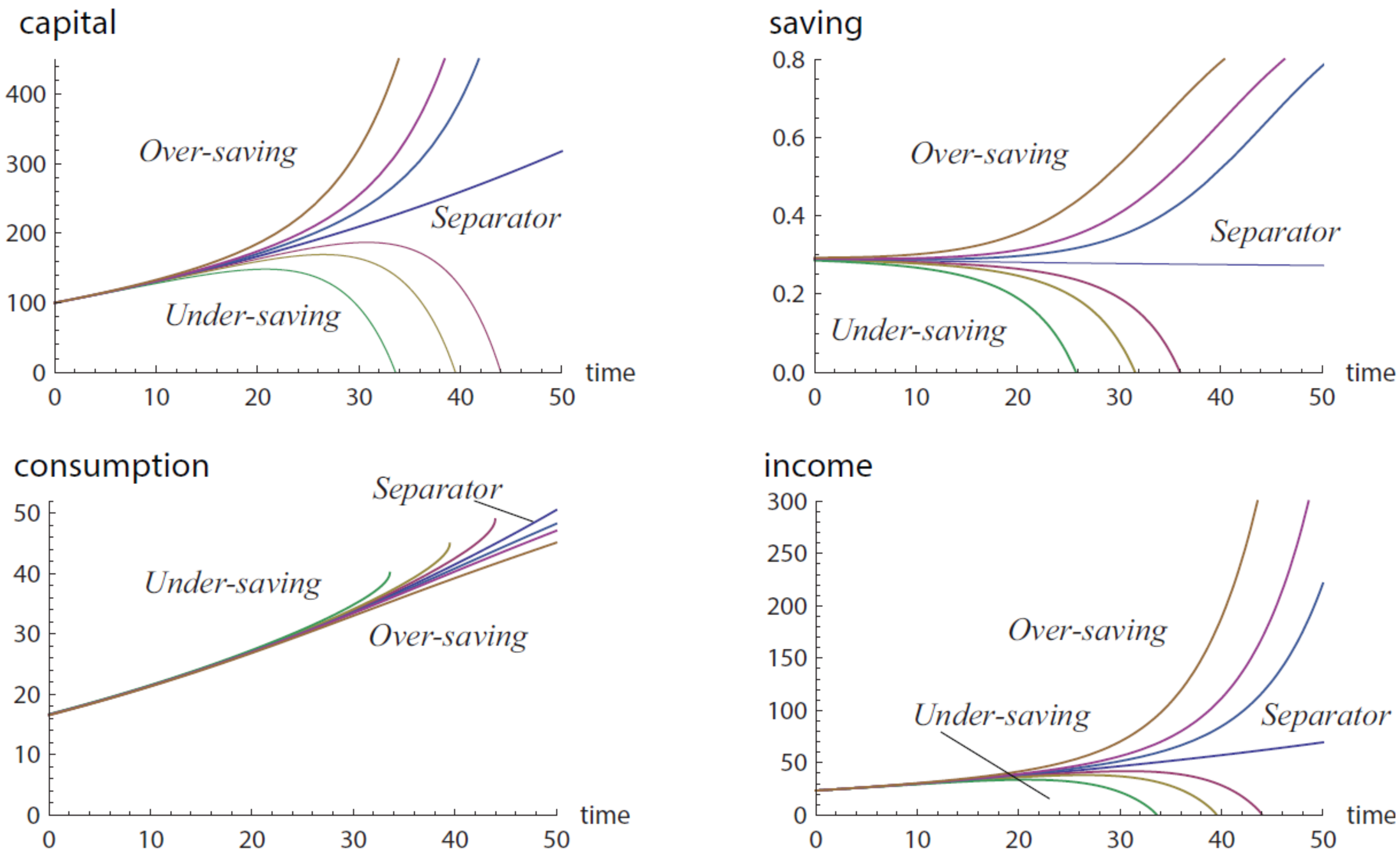

Figure 5 : The **time paths** for: $k(t)$, $c(t)$, $y(t)$, and $s(t)$. **CES**-*baseline case.*

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Solutions and Phase Portrait with CIES and CES

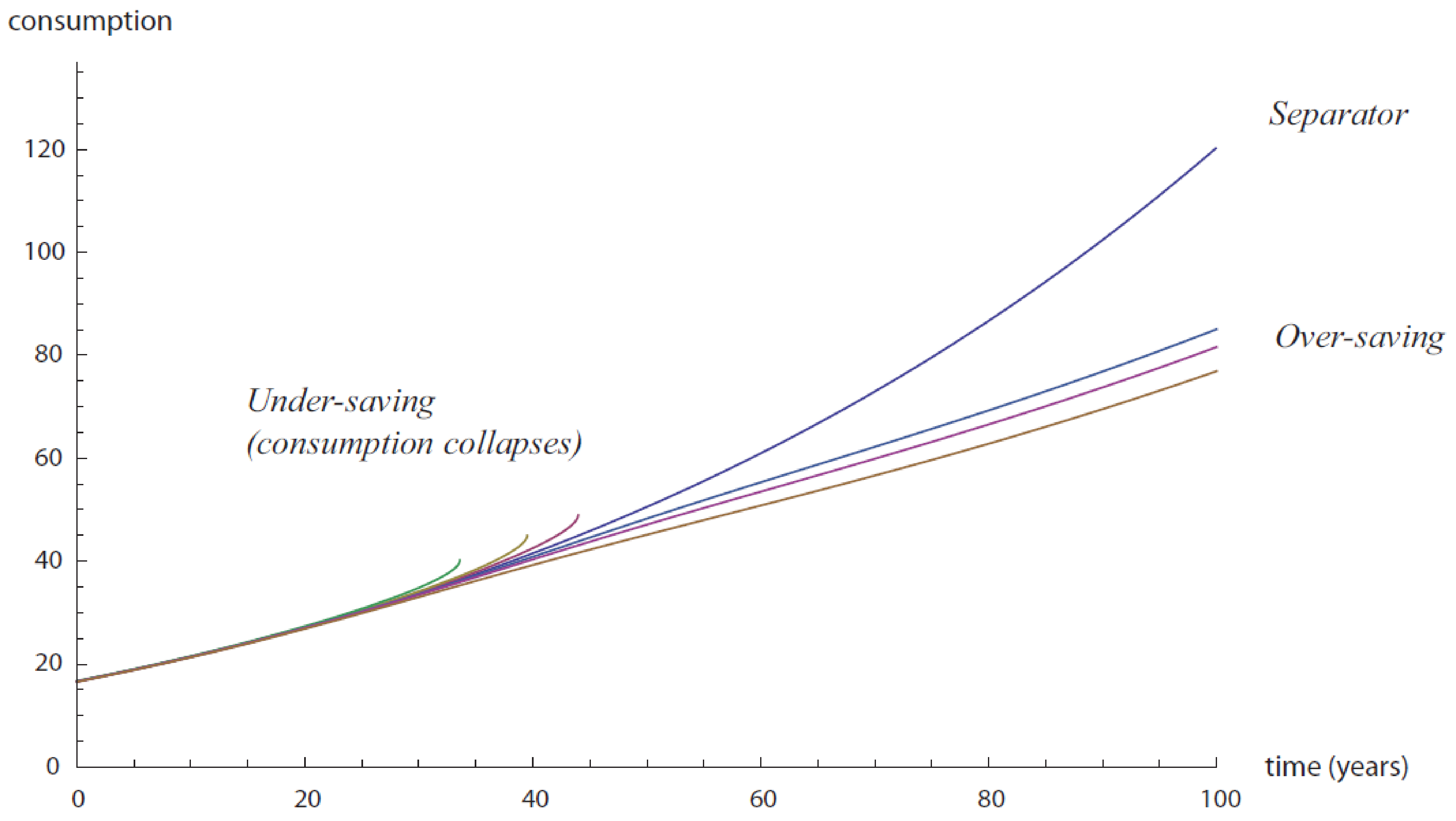

Figure 6 : The time paths for: $c(t)$, **short run** and **long run**, CES-baseline case

Parameter Sensitivity of Optimal Time Paths

Parameter sets	CES-baseline	high θ	low θ	high σ,	low σ
θ	1.000	3.000	0.750	1.000	1.000
ρ	0.160	0.140	0.163	0.160	0.160
σ	2.000	2.000	2.000	2.500	1.500
γ	0.661	0.661	0.661	0.540	1.200

Table 2. The **Parameter Sets** used in **Sensitivity Analyses**.

θ Relative Risk Aversion

η Intertemporal elasticity of substitution : $\eta = 1/\theta$

θ : 0.5 – 5.0

η : 2.0 – 0.5

$\theta = 1 = \eta : u(c) = ln\, c$

Asymptotic equivalence (t $= \infty$), $\underline{b} = 0.2$, $\underline{\widehat{y}} = 0.01$, $s^* = 0.25$:

$$\underline{b} = \gamma\, a^{\frac{\sigma}{\sigma-1}}\ ;\ \underline{\widehat{y}} = \frac{\underline{b} - \delta - \rho}{\theta}\ ;\ s^* = \frac{(\underline{b} - \delta - \rho)/\theta + n + \delta}{\underline{b}} \equiv \frac{\underline{\widehat{y}} + n + \delta}{\underline{b}}$$

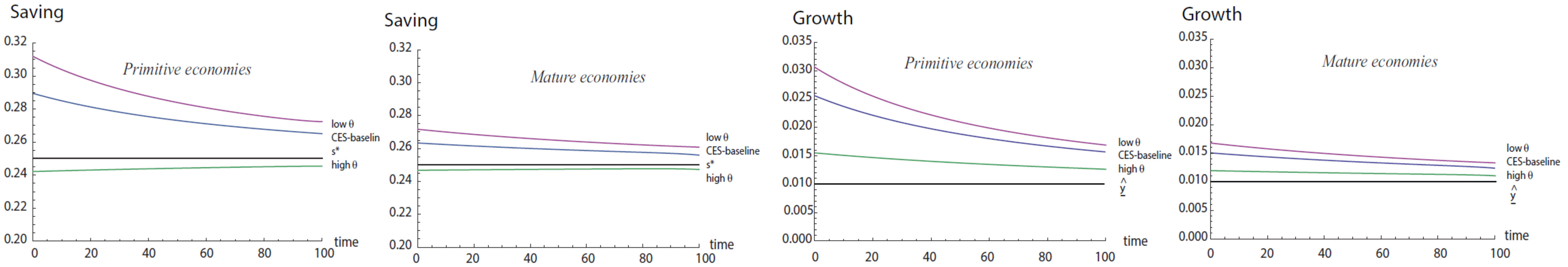

Figure 7 : Low θ **/high** ρ, Table 2. **Time paths** for $s(t)$ and $\widehat{y}(t)$ in **primitive** ($k(0) = 100$) and **mature** ($k(0) = 1000$) economies.

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Parameter Sensitivity of Optimal Time Paths

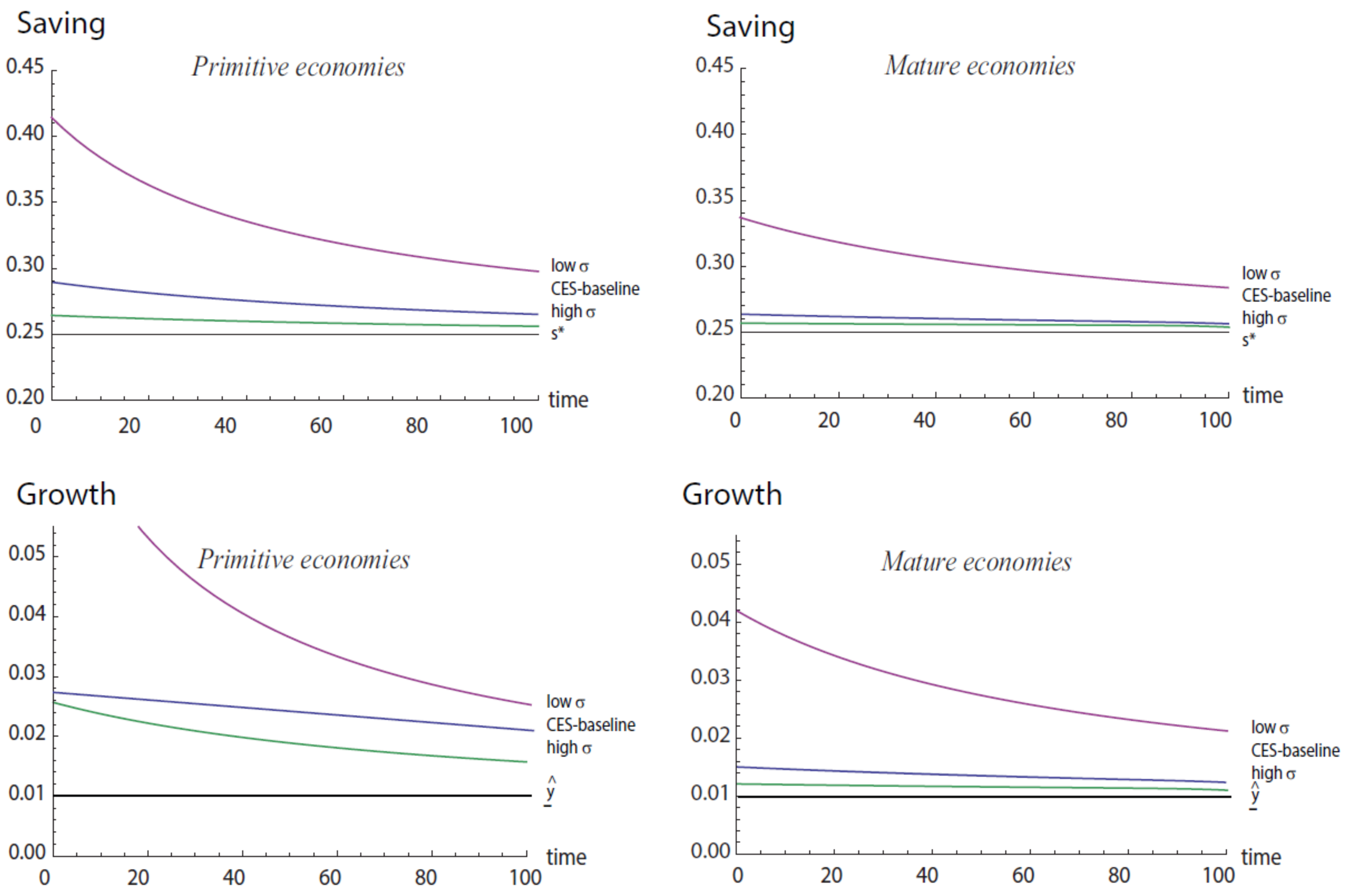

Figure 8 : High γ**/low** σ, Table 2. **Time paths** for $s(t)$ and $\widehat{y}\,(t)$ in **primitive** ($k(0) = 100$) and **mature** ($k(0) = 1000$) economies.

Phase Portraits with the Extended CD Function

$$Y = F(L,K) = AK + BK^{\alpha} L^{1-\alpha} \,;\; A,\, B, > 0\,,\, 0 < \alpha < 1$$

$$\sigma(k) \equiv -\frac{[f(k) - f'(k)\,k]\,f'(k)}{f''(k)\,k\,f(k)} = \frac{\alpha B + A\,k^{1-\alpha}}{\alpha B + \alpha\,A\,k^{1-\alpha}}$$

$$\lim_{k\to 0} \sigma(k) = 1\,; \qquad \lim_{k\to\infty} \sigma(k) = 1/\alpha$$

- The **Ramsay dynamic system** becomes with extended CD and CIES

$$\dot{k} = Ak + Bk^{\alpha} - c - (n+\delta)k$$

$$\dot{c} = (c/\theta)[A + \alpha B k^{\alpha-1} - \delta - \rho]$$

- The **transformed dynamic system** becomes

$$\dot{z} = z\,(1-\alpha)(A/z - 1)(z - x - n - \delta)\,,$$

$$\dot{x} = x\,[\,x - D - \frac{\theta - \alpha}{\theta} \cdot (z - A)\,]$$

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Solutions and Phase Portrait with CIES and extended CD

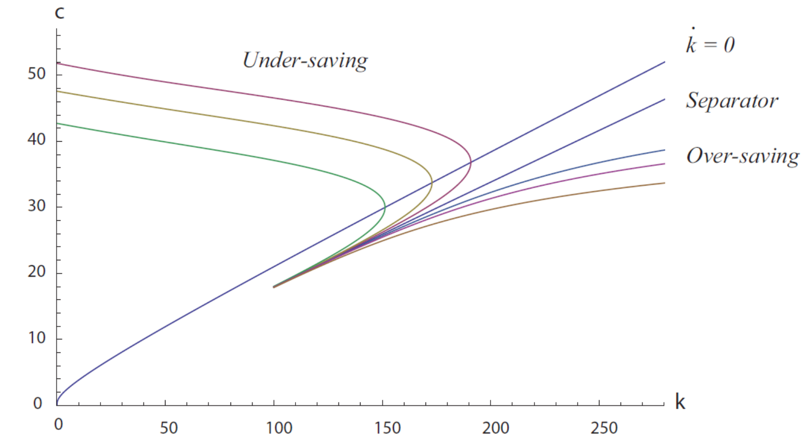

Figure 9 : The **phase portrait** (orbits) in the **state** (k, c) **space**. *Extended CD.*

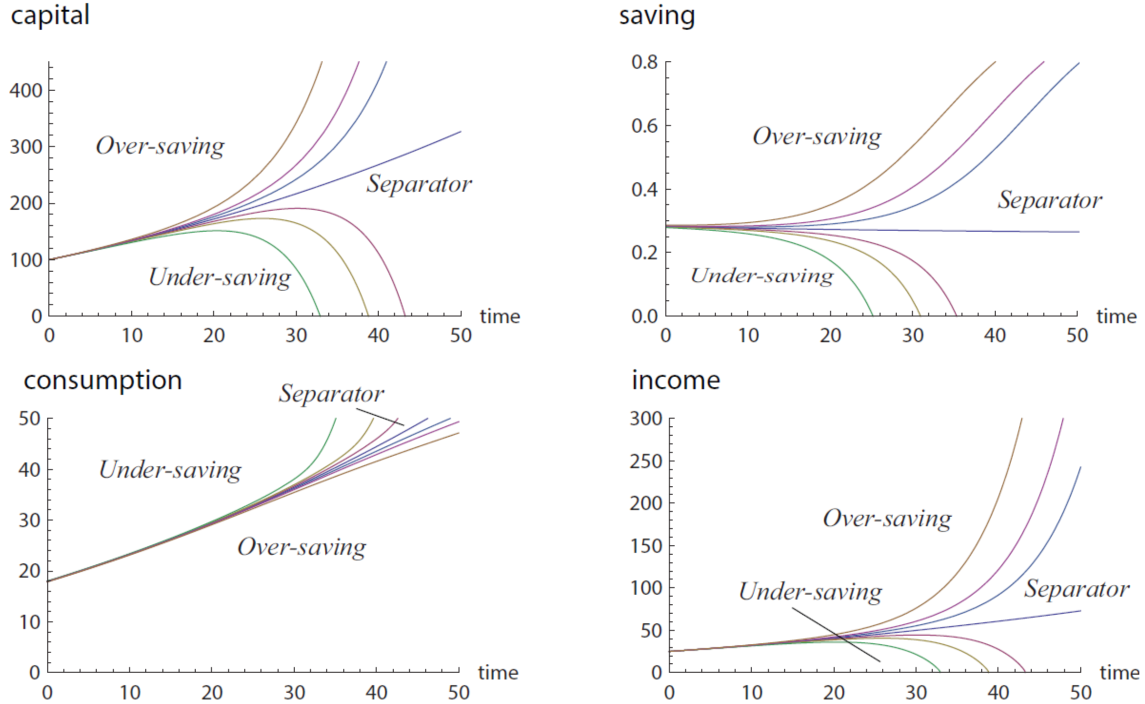

Figure 10 : The **time paths** for: $k(t)$, $c(t)$, $y(t)$, and $s(t)$. *Extended CD.*

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The Ramsey Model with CES function : $\sigma = \infty$

$$Y = F(L, K) = \gamma [(1 - a)L + aK] \; ; \; y = f(k)$$

- The **Ramsay dynamic system :** *Optimal control* system is *Affine dynamics*

$$\dot{k} = \gamma [1 - a + ak] - (n + \delta) k - c = \gamma(1 - a) + (\gamma a - \delta - n) k - c$$

$$\dot{c} = (1/\theta) [\gamma a - \delta - \rho] c \quad > 0 \; ; \; \dot{k} = 0 : \; c = \gamma(1 - a) + (\gamma a - \delta - n) k$$

- The **transformed non-linear dynamic system** of the *Affine dynamics* becomes :

$$\dot{z} = [a \gamma - z] [z - x - n - \delta]$$

$$\dot{x} = x [x - z + \frac{\gamma a - \delta - \rho}{\theta} + n + \delta]; \quad node : (z^*, x^* = 0)$$

- *Saddle point* of transformed system with *asymptotic saving/growth rates*, are :

$$z^* = \underline{b} = \gamma a \; ; \; x^* = z^* [\frac{\theta - 1}{\theta}] + \frac{\delta + \rho}{\theta} - n - \delta = [\gamma a - \delta] [\frac{\theta - 1}{\theta}] + \frac{\rho}{\theta} - n$$

$$s^* = 1 - \frac{x^*}{z^*} = \frac{(\gamma a - \delta - \rho)/\theta + n + \delta}{\gamma a} = \frac{\underline{\widehat{y}} + n + \delta}{\gamma a}$$

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The Ramsey Model with CES function : $f(k) = \gamma a k$

$$y = f(k) = \gamma a k \quad \text{- AK-model, } B = 0$$

- The **Ramsay dynamic system :** *Optimal control* system is *Linear dynamics*

$$\dot{k} = (\gamma a - \delta - n)\, k - c \equiv a_{11}\, k + a_{12}\, c\,; \quad a_{11} = 0.16\,, \; a_{12} = -1$$

$$\dot{c} = (1/\theta)\,[\gamma a - \delta - \rho]\, c \equiv a_{22}\, c \quad ; \quad a_{21} = 0\,, \; a_{22} = 0.01$$

using CES-baseline *parameters :* $\underline{b} = \gamma\, a = 0.20,\ \delta = 0.03,\ \rho = 0.16,\ n = 0.01,\ \theta = 1$

- The *triangular coefficient matrix* implies that,

$$\forall t\ :\ \dot{c}(t)/c(t) = c(t) = (1/\theta)\,[\underline{b} = -\delta - \rho] = \underline{\widehat{y}} = a_{22}\ ; \quad a_{22} = 0.01$$

- The isocline, $\dot{k} = 0$, as a straight line :

$$\dot{k} = 0\ :\ c = (\gamma a - \delta - n)\, k = a_{11}\, k\ ; \quad a_{11} = 0.16$$

The Ramsey Model with Closed Form Solutions

- **Ray ("eigen-vector")** with *slope* - α_1 - is the *trajectory* for the **separator**

$$\forall t \;:\; c^*(t)/k^*(t) = \alpha_1 = a_{11} - a_{22} = c_0^*/k_0 = 0.15/1.0 = 15/100 = 0.15 = x^*$$

- The *unique separator, optimal time paths,* $\varphi^*(t)$, along the ***trajectory*** are expressed by :

$$\forall t \;\; \varphi^*(t) \equiv [\,k^*(t),\, c^*(t)\,] = e^{a_{22}\,t}\,[\,k_0,\, c_0^*\,] = k_0\, e^{a_{22}\,t}\,[\,1,\, \alpha_1\,]\,,\; k_0 = 1\,;\; k_0 = 100$$

- The *constant* (time invariant) **optimal** *saving rates* $s^*(t)$ along the ***separator*** is :

$$\forall t \;\; s^*(t) = 1 - c^*(t)/y^*(t) = 1 - \alpha_1\, k^*(t)/\gamma\, a\, k^*(t) = 1 - \frac{\alpha_1}{\gamma\, a} = 1 - \frac{x^*}{z^*} = 0.25$$

- All time paths $\varphi(t)$ – *General Solution* $[k(t),\, c(t)]$ – have the *closed form* :

$$\varphi(t) \;:\; k(t) = k_0\, e^{a_{11}\,t} + \frac{c_0}{a_{11} - a_{22}}\left(e^{a_{22}\,t} - e^{a_{11}\,t} \right) \;;\; c(t) = c_0\, e^{a_{22}\,t} \;; \quad k_0 = 100$$

- Time paths for $k(t)/c(t)$ and some *non-optimal saving paths* $s(t)$, are given by :

$$k(t)/c(t) = \frac{1}{a_{11} - a_{22}} + \left(\frac{k_0}{c_0} - \frac{1}{a_{11} - a_{22}} \right) e^{(a_{11} - a_{22})\,t} \;; \quad k_0 = 100$$

$$s(t) = 1 - c(t)/y(t) = 1 - c(t)/\gamma\, a\, k(t) = 1 - \frac{1}{\gamma\, a}\,\frac{c(t)}{k(t)} \;; \quad k_0 = 100$$

- In *eigen-vector space*, the *phase portraits* of the *General Solution* is called an *unstable node*, Pontrygain (1962, p.117), Birkoff & Rota (1989, p.148) and Arnold (1973, p.118)

Solutions and Phase Portrait - CIES and Linear Dynamics

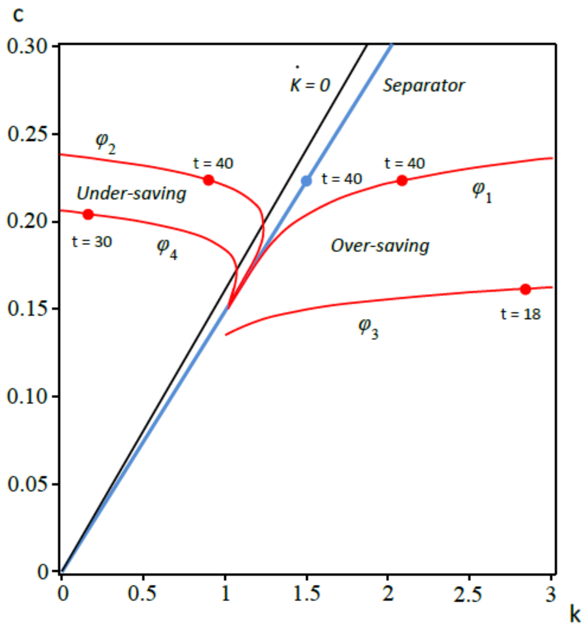

Figure 11.a

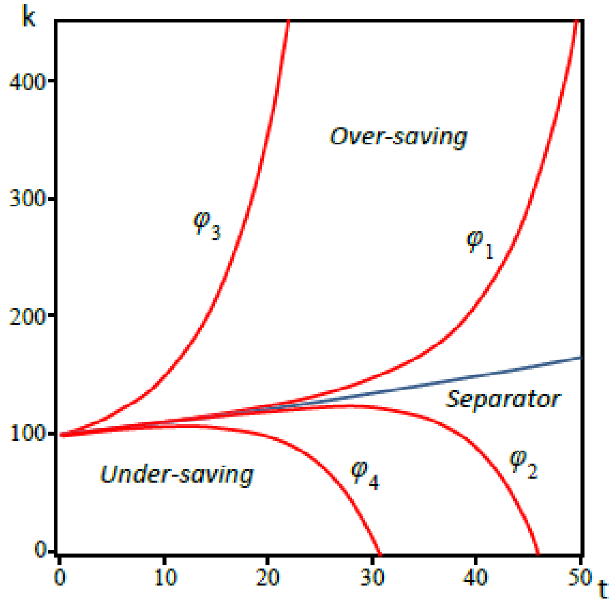

Figure 11.b

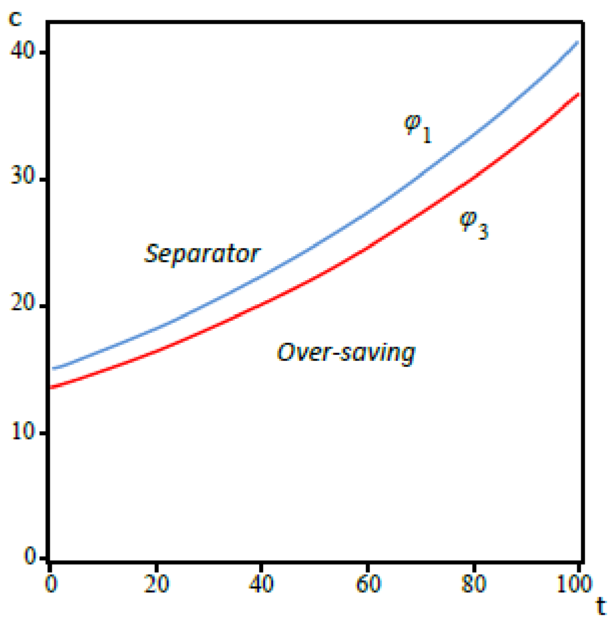

Figure 11.c

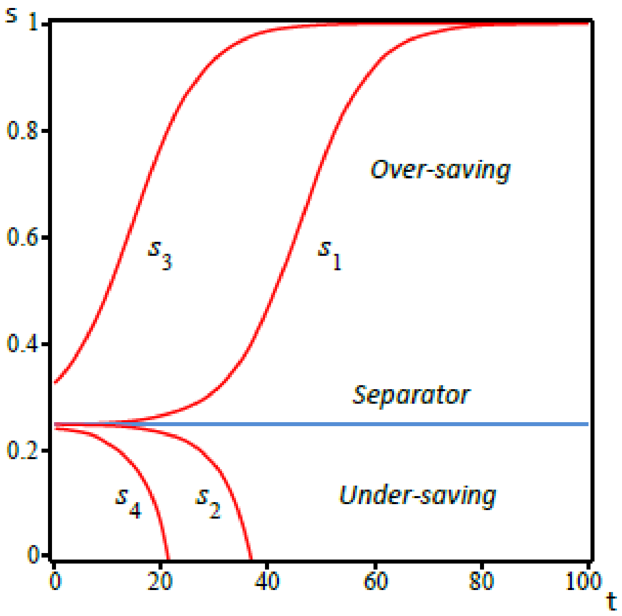

Figure 11.d

Figure 11a. The **Phase Portrait** – $\varphi(t)$ – in the state $(k,\ c)$ space ; $k(0) = 1$ with *initial* (control) values : $c_0^* = 0{,}15$, separator ; *Fig. b-d* : $k(0) = 100$: $c_0^* = 15$,

$c_0 = 15 \cdot 0.999 = 14.985,\ \varphi_1(t)$; $c_0 = 15 \cdot 0.9 = 13.5,\ \varphi_3(t)$;

$c_0 = 15 \cdot 1.001 = 15.015,\ \varphi_2(t)$; $c_0 = 15 \cdot 1.01 = 15.15,\ \varphi_4(t)$.

Figure 11b. Ramsey Model Time paths for $k(t)$: $\varphi(t)$ – short run.

Figure 11c. Ramsey Model Time paths for $c(t)$: $\varphi(t)$ – long run.

Figure 11d. Ramsey Model Time paths for $s(t)$: $\varphi(t)$ – short run and long run.

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Conclusion

We have ***derived*** and ***computed*** the *optimal saving solutions* for optimal economic growth generated endogenously by a ***Ramsey model***. *Sufficient conditions* are presented for *persistent* economic growth within a ***"standard parametrically unified Ramsey model"***.

In ***phase diagrams*** of trajectories (solutions), the ***optimal path (trajectory)*** is a ***separator***. *Below* the separator, *over-saving* diminishes consumption, *ultimately* leading to a sub-optimal situation where *all incomes* are *saved*. *Above* the separator *under-saving* suddenly *collapses* the economy as its productive *capital* vanishes *to zero*.

Comprehensive *numerical applications* for *CIES preferences* and *CES technologies*, together with *parametric sensitivity* analyses of the *optimal solutions* were ***demonstrated***.

Final Comments and Suggestions

- Asymptotic "real interest rates" ($\underline{b}$) – Asymptotic growth rate ($\underline{\widehat{y}}$) – Saving rates (s^*), are :

$$\underline{b} = \gamma\, a^{\frac{\sigma}{\sigma-1}} \;;\; \underline{\widehat{y}} = \frac{\underline{b} - \delta - \rho}{\theta} \;;\; s^* = \frac{(\underline{b} - \delta - \rho)/\theta + n + \delta}{\underline{b}} \equiv \frac{\underline{\widehat{y}} + n + \delta}{\underline{b}}$$

- Benchmark solutions were : $\underline{b} = 0.2,\ \underline{\widehat{y}} = 0.01,\ s^* = 0.25$

- Existent conditions for the optimal solutions – separator – were :

$$\underline{b} - \delta > \rho > (\underline{b} - \delta + n\,\theta)/(1+\theta) > n$$

- We used the additive intertemporal cardinal utility function :

$$u\,[\,c(t)\,]\,e^{-\rho t} \equiv I(t),\ (\text{Integrand})$$

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Thank you for your Attention