Coverings of Spectral Triples

December 12, 2017

Petr R. Ivankov*
e-mail: * monster.ivankov@gmail.com

It is well-known that any covering space of a Riemannian manifold has the natural structure of a Riemannian manifold. This article contains a noncommutative generalization of this fact. Since any Riemannian manifold with a Spin-structure defines a spectral triple, the spectral triple can be regarded as a noncommutative Spin-manifold. Similarly there is an algebraic construction which is a noncommutative generalization of topological covering. This article contains a construction of spectral triple on the "noncommutative covering space".

Contents

1 Motivation. Preliminaries

1.1 Prototype. Coverings of Riemannian manifolds

This article proves a noncommutative generalization of the following proposition.

Proposition 1.1. *(Proposition 5.9 [14])*

1. *Given a connected manifold M there is a unique (unique up to isomorphism) universal covering manifold, which will be denoted by $\widetilde{M}$.*

2. *The universal covering manifold $\widetilde{M}$ is a principal fibre bundle over M with group $\pi_1(M)$ and projection $p : \widetilde{M} \to M$, where $\pi_1(M)$ is the first homotopy group of M.*

3. *The isomorphism classes of covering spaces over M are in 1:1 correspondence with the conjugate classes of subgroups of $\pi_1(M)$. The correspondence is given as follows. To each subgroup H of $\pi_1(M)$, we associate $E = \widetilde{M}/H$. Then the covering manifold E corresponding to H is a fibre bundle over M with fibre $\pi_1(M)/H$ associated with the principal bundle $\widetilde{M}(M, \pi_1(M))$. If H is a normal subgroup of $\pi_1(M)$, $E = \widetilde{M}/H$ is a principal fibre bundle with group $\pi_1(M)/H$ and is called a regular covering manifold of M.*

1.2. If $\widetilde{M}$ is a covering space of Riemannian manifold M then it is possible to give $\widetilde{M}$ a Riemannian structure such that $\pi : \widetilde{M} \to M$ is a local isometry (this metric is called the *covering metric*). cf. [9] for details.

Gelfand-Naǐmark theorem [2] states the correspondence between locally compact Hausdorff topological spaces and commutative C^*-algebras.

Theorem 1.3. *[2] (Gelfand-Naǐmark). Let A be a commutative C^*-algebra and let $\mathcal{X}$ be the spectrum of A. There is the natural $*$-isomorphism $\gamma : A \to C_0(\mathcal{X})$.*

So any (noncommutative) C^*-algebra may be regarded as a generalized (noncommutative) locally compact Hausdorff topological space. Articles [12, 18] contain noncommutative analogs of coverings. The spectral triple [11, 21] can be regarded as a noncommutative generalization of Riemannian manifold. Having analogs of both coverings and Riemannian manifolds one can proof a noncommutative generalization of the Proposition 1.1.

Following table contains a list of special symbols.

Symbol	Meaning
A^G	Algebra of G-invariants, i.e. $A^G = \{a \in A \mid ga = a, \forall g \in G\}$
$\mathrm{Aut}(A)$	Group * - automorphisms of C^* algebra A
$B(\mathcal{H})$	Algebra of bounded operators on a Hilbert space $\mathcal{H}$
$\mathbb{C}$ (resp. $\mathbb{R}$)	Field of complex (resp. real) numbers
$C(\mathcal{X})$	C^* - algebra of continuous complex valued functions on a space $\mathcal{X}$
$C_0(\mathcal{X})$	C^* - algebra of continuous complex valued functions on a topological space $\mathcal{X}$ equal to 0 at infinity
$C_c(\mathcal{X})$	Algebra of continuous complex valued functions on a topological space $\mathcal{X}$ with compact support
$G(\widetilde{\mathcal{X}} \mid \mathcal{X})$	Group of covering transformations of covering projection $\widetilde{\mathcal{X}} \to \mathcal{X}$ [20]
$\mathcal{H}$	Hilbert space
$\mathcal{K} = \mathcal{K}(\mathcal{H})$	C^* - algebra of compact operators
$K(A)$	Pedersen ideal of C^*-algebra A
$\varinjlim$	Direct limit
$\varprojlim$	Inverse limit
$M(A)$	A multiplier algebra of C^*-algebra A
$\mathbb{M}_n(A)$	The $n \times n$ matrix algebra over C^*- algebra A
$\mathbb{N}$	A set of positive integer numbers
$\mathbb{N}^0$	A set of nonnegative integer numbers
$\overline{G/G'} \subset G$	A set of representatives of a quotient group G/G'
S^n	The n-dimensional sphere
$\mathbb{Q}$	Field of rational numbers
$\mathbb{Z}$	Ring of integers
$\mathbb{Z}_n$	Ring of integers modulo n
$\overline{k} \in \mathbb{Z}_n$	An element in $\mathbb{Z}_n$ represented by $k \in \mathbb{Z}$
$X \backslash A$	Difference of sets $X \backslash A = \{x \in X \mid x \notin A\}$
$\lvert X \rvert$	Cardinal number of the finite set
$f\lvert_{A'}$	Restriction of a map $f : A \to B$ to $A' \subset A$, i.e. $f\lvert_{A'} : A' \to B$

1.2 Topology

1.2.1 Coverings

Definition 1.4. [20] Let $\widetilde{\pi} : \widetilde{\mathcal{X}} \to \mathcal{X}$ be a continuous map. An open subset $\mathcal{U} \subset \mathcal{X}$ is said to be *evenly covered* by $\widetilde{\pi}$ if $\widetilde{\pi}^{-1}(\mathcal{U})$ is the disjoint union of open subsets of $\widetilde{\mathcal{X}}$ each of which is mapped homeomorphically onto $\mathcal{U}$ by $\widetilde{\pi}$. A continuous map $\widetilde{\pi} : \widetilde{\mathcal{X}} \to \mathcal{X}$ is called a *covering projection* if each point $x \in \mathcal{X}$ has an open neighborhood evenly covered by $\widetilde{\pi}$. $\widetilde{\mathcal{X}}$ is called the *covering space* and $\mathcal{X}$ the *base space* of the covering.

Definition 1.5. [20] A fibration $p : \widetilde{\mathcal{X}} \to \mathcal{X}$ with unique path lifting is said to be *regular* if, given any closed path ω in $\mathcal{X}$, either every lifting of ω is closed or none is closed.

Definition 1.6. [20] A topological space $\mathcal{X}$ is said to be *locally path-connected* if the path components of open sets are open.

Denote by π_1 the functor of fundamental group [20].

Theorem 1.7. *[20] Let $p : \widetilde{\mathcal{X}} \to \mathcal{X}$ be a fibration with unique path lifting and assume that a nonempty $\widetilde{\mathcal{X}}$ is a locally path-connected space. Then p is regular if and only if for some $\widetilde{x}_0 \in \widetilde{\mathcal{X}}$, $\pi_1(p)\,\pi_1\left(\widetilde{\mathcal{X}}, \widetilde{x}_0\right)$ is a normal subgroup of $\pi_1(\mathcal{X}, p(\widetilde{x}_0))$.*

Definition 1.8. [20] Let $p : \widetilde{\mathcal{X}} \to \mathcal{X}$ be a covering. A self-equivalence is a homeomorphism $f : \widetilde{\mathcal{X}} \to \widetilde{\mathcal{X}}$ such that $p \circ f = p$. This group of such homeomorphisms is said to be the *group of covering transformations* of p or the *covering group*. Denote by $G\left(\widetilde{\mathcal{X}} \mid \mathcal{X}\right)$ this group.

Proposition 1.9. *[20] If $p : \widetilde{\mathcal{X}} \to \mathcal{X}$ is a regular covering and $\widetilde{\mathcal{X}}$ is connected and locally path connected, then $\mathcal{X}$ is homeomorphic to space of orbits of $G\left(\widetilde{\mathcal{X}} \mid \mathcal{X}\right)$, i.e. $\mathcal{X} \approx \widetilde{\mathcal{X}}/G\left(\widetilde{\mathcal{X}} \mid \mathcal{X}\right)$. So p is a principal bundle.*

Corollary 1.10. *[20] Let $p : \widetilde{\mathcal{X}} \to \mathcal{X}$ be a fibration with a unique path lifting. If $\widetilde{\mathcal{X}}$ is connected and locally path-connected and $\widetilde{x}_0 \in \widetilde{\mathcal{X}}$ then p is regular if and only if $G\left(\widetilde{\mathcal{X}} \mid \mathcal{X}\right)$ transitively acts on each fiber of p, in which case*

$$\psi : G\left(\widetilde{\mathcal{X}} \mid \mathcal{X}\right) \approx \pi_1(\mathcal{X}, p(\widetilde{x}_0)) / \pi_1(p)\,\pi_1\left(\widetilde{\mathcal{X}}, \widetilde{x}_0\right).$$

Remark 1.11. Above results are copied from [20]. Below the *covering projection* word is replaced with *covering*.

1.2.2 Vector bundles

We refer to [13] for a notion of *(locally trivial) vector bundle* with base $\mathcal{X}$ and an *inverse image* of a vector bundle. For any topological space $\mathcal{X}$ there is a category $\mathrm{Vect}(\mathcal{X})$ of vector bundles with base $\mathcal{X}$. If $f : \mathcal{X} \to \mathcal{Y}$ is a continuous map then there is an *inverse image* *functor* $f^*: \mathrm{Vect}(\mathcal{Y}) \to \mathrm{Vect}(\mathcal{X})$ (cf. [13]). Let $\pi : \widetilde{\mathcal{X}} \to \mathcal{X}$ be a covering projection, and let $E \in \mathrm{Vect}\,(\mathcal{X})$, $\widetilde{E} = \pi^* E$ is an *inverse image*. Any $\mathbb{C}$-(anti)linear map φ from $\Gamma(\mathcal{X}, E)$ (resp. dense $\mathbb{C}$-subspace of $X \subset \Gamma(\mathcal{X}, E)$) to $\Gamma(\mathcal{X}, E)$ naturally induces a $\mathbb{C}$ (anti)linear map φ^* from $\Gamma\left(\widetilde{\mathcal{X}}, \widetilde{E}\right)$ (resp. dense $\mathbb{C}$-subspace of $\widetilde{X} \subset \Gamma\left(\widetilde{\mathcal{X}}, \widetilde{E}\right)$) to $\Gamma\left(\widetilde{\mathcal{X}}, \widetilde{E}\right)$.

1.12. Let $\pi : \widetilde{\mathcal{X}} \to \mathcal{X}$ be a covering projection, and let $E \in \mathrm{Vect}\,(\mathcal{X})$, $\widetilde{E} = \pi^* E$ be an inverse image. If $\widetilde{\mathcal{U}} \subset \widetilde{\mathcal{X}}$ is an open subset such that a restriction $\pi|_{\mathcal{U}} : \widetilde{\mathcal{U}} \to \pi(\widetilde{\mathcal{U}}) = \mathcal{U}$ is a homeomorphism. Then there are *-isomorphism $C(\widetilde{\mathcal{U}}) \xrightarrow{\approx} C\,(\mathcal{U})$ and isomorphism $\Gamma\left(\widetilde{\mathcal{U}}, \widetilde{E}|_{\widetilde{\mathcal{U}}}\right) \approx \Gamma\,(\mathcal{U}, E_{\mathcal{U}})$. If $X = \Gamma\,(\mathcal{X}, E)$ or $\widetilde{X} \subset \Gamma\left(\widetilde{\mathcal{X}}, \widetilde{E}\right)$ is a dense subspace, $\varphi : X \to \Gamma\,(\mathcal{X}, E)$ is $\mathbb{C}$ (anti)linear map, and then there is a following commutative diagram

$$\begin{array}{ccc}
\widetilde{X}|_{\widetilde{\mathcal{U}}} & \xrightarrow{\varphi^*|_{\widetilde{\mathcal{U}}}} & \Gamma\left(\widetilde{\mathcal{U}}, \widetilde{E}|_{\widetilde{\mathcal{U}}}\right) \\
\approx \downarrow & & \downarrow \approx \\
X|_{\mathcal{U}} & \xrightarrow{\varphi|_{\mathcal{U}}} & \Gamma\left(\mathcal{U}, \widetilde{E}|_{\mathcal{U}}\right)
\end{array}$$

Definition 1.13. The map φ^* is said to be an *inverse image* or *lift* of φ.

1.14. Let $\mathcal{X}$ be a topological space and S the complex linear bundle on $\mathcal{X}$. Suppose that for any $x \in \mathcal{X}$ there is the scalar product $(\cdot, \cdot)_x : S_x \times S_x \to \mathbb{C}$ and there is a measure $\mu_{\mathcal{X}}$ on $\mathcal{X}$. If $\Gamma\,(M, S)$ is the space of continuous sections of S then we suppose that for any $\xi, \eta \in \Gamma\,(M, S)$ the map $\mathcal{X} \to \mathbb{C}$ given by $x \mapsto (\xi_x, \eta_x)_x$ is continuous. There is the scalar product $(\cdot, \cdot) : \Gamma\,(M, S) \times \Gamma\,(M, S) \to \mathbb{C}$ given by

$$(\xi, \eta) = \int_{\mathcal{X}} (\xi_x, \eta_x)\, d\, \mu_{\mathcal{X}}$$

Denote by $L^2\,(\mathcal{X}, S, \mu_{\mathcal{X}})$ or $L^2\,(\mathcal{X}, S)$ the Hilbert norm completion of $\Gamma\,(M, S)$. There is the natural representation

$$C_0\,(\mathcal{X}) \to B\left(L^2\,(\mathcal{X}, S)\right). \tag{1.1}$$

Definition 1.15. In the situation of 1.14 we say that S is *Hermitian vector bundle*.

1.2.3 Locally compact spaces

In this article we consider second-countable locally compact Hausdorff spaces only [16]. So we will say a "topological space" (resp. "compact space") instead "locally compact second-countable Hausdorff space" (resp. "compact second-countable Hausdorff space"). This subsection contains well known facts, I follow to [16].

There are two equivalent definitions of $C_0\,(\mathcal{X})$ and both of them are used in this article.

Definition 1.16. An algebra $C_0(\mathcal{X})$ is the norm closure of the algebra $C_c(\mathcal{X})$ of compactly supported continuous functions.

Definition 1.17. A C^*-algebra $C_0(\mathcal{X})$ is given by the following equation

$$C_0(\mathcal{X}) = \{\varphi \in C_b(\mathcal{X}) \mid \forall \varepsilon > 0 \;\; \exists K \subset \mathcal{X} \; (K \text{ is compact}) \;\&\; \forall x \in \mathcal{X} \backslash K \; |\varphi(x)| < \varepsilon\},$$

i.e.

$$\left\| \varphi|_{\mathcal{X} \backslash K} \right\| < \varepsilon.$$

Theorem 1.18. *[16] Every compact Hausdorff space is normal.*

Theorem 1.19. *[16]* ***Urysohn lemma.*** *Let $\mathcal{X}$ be a normal space, let $\mathcal{A}$, $\mathcal{B}$ be disjoint closed subsets of $\mathcal{X}$. Let $[a, b]$ be a closed interval in the real line. Then there exist a continuous map $f : \mathcal{X} \to [a, b]$ such that $f(\mathcal{A}) = \{a\}$ and $f(\mathcal{B}) = \{b\}$.*

Theorem 1.20. *[16]* ***Urysohn metrization theorem.*** *Every regular space with a countable basis is metrizable.*

From the Theorems 1.18 and 1.19 it follows that if $\mathcal{X}$ is locally compact Hausdorff space $x \in \mathcal{X}$, and $\mathcal{B}$ is closed subset of $\mathcal{X}$, such that $x \notin \mathcal{B}$ then there exist a continuous map $f : \mathcal{X} \to [a, b]$ such that $f(x) = a$ and $f(\mathcal{B}) = \{b\}$. It means that locally compact Hausdorff space is completely regular, whence $\mathcal{X}$ is regular (cf. [16]), and from the Theorem 1.20 it follows next corollary.

Corollary 1.21. *Every locally compact second-countable Hausdorff space is metrizable.*

Theorem 1.22. *[16] Every metrizable space is paracompact.*

Definition 1.23. [16] If $\phi : \mathcal{X} \to \mathbb{C}$ is continuous then the *support* of ϕ is defined to be the closure of the set $\phi^{-1}(\mathbb{C} \backslash \{0\})$ Thus if x lies outside the support, there is some neighborhood of x on which ϕ vanishes. Denote by $\mathfrak{supp}\,\phi$ the support of ϕ.

Definition 1.24. [16] Let $\{\mathcal{U}_\alpha \in \mathcal{X}\}_{\alpha \in J}$ be an indexed open covering of $\mathcal{X}$. An indexed family of functions

$$\phi_\alpha : \mathcal{X} \to [0, 1]$$

is said to be a *partition of unity* , dominated by $\{\mathcal{U}_\alpha\}_{\alpha \in J}$, if:

1. $\phi_\alpha(\mathcal{X} \backslash \mathcal{U}_\alpha) = \{0\}$
2. The family $\{\mathfrak{supp}(\phi_\alpha) = \mathfrak{cl}(\{x \in \mathcal{X} \mid \phi_\alpha > 0\})\}$ is locally finite.
3. $\sum_{\alpha \in J} \phi_\alpha(x) = 1$ for any $x \in \mathcal{X}$.

Theorem 1.25. *[16] Let $\mathcal{X}$ be a paracompact Hausdorff space; let $\{\mathcal{U}_\alpha \in \mathcal{X}\}_{\alpha \in J}$ be an indexed open covering of $\mathcal{X}$. Then there exists a partition of unity, dominated by $\{\mathcal{U}_\alpha\}$.*

1.3 Inverse limits of coverings

This subsection is concerned with a topological construction of the inverse limit in the category of coverings.

Definition 1.26. The sequence of regular finite-fold coverings

$$\mathcal{X} = \mathcal{X}_0 \leftarrow ... \leftarrow \mathcal{X}_n \leftarrow ...$$

is said to be a *(topological) finite covering sequence* if following conditions hold:

- The space $\mathcal{X}_n$ is a second-countable [16] locally compact connected Hausdorff space for any $n \in \mathbb{N}^0$,
- If $k < l < m$ are any nonnegative integer numbers then there is the natural exact sequence
$$\{e\} \to G\left(\mathcal{X}_m \mid \mathcal{X}_l\right) \to G\left(\mathcal{X}_m \mid \mathcal{X}_k\right) \to G\left(\mathcal{X}_l \mid \mathcal{X}_k\right) \to \{e\}.$$

For any finite covering sequence we will use a following notation

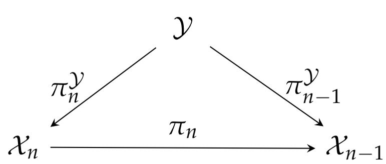
$$\mathfrak{S} = \{\mathcal{X} = \mathcal{X}_0 \leftarrow ... \leftarrow \mathcal{X}_n \leftarrow ...\} = \{\mathcal{X}_0 \leftarrow ... \leftarrow \mathcal{X}_n \leftarrow ...\}, \quad \mathfrak{S} \in \mathfrak{FinTop}.$$

Definition 1.27. Let $\{\mathcal{X} = \mathcal{X}_0 \leftarrow ... \leftarrow \mathcal{X}_n \leftarrow ...\} \in \mathfrak{FinTop}$, and let $\widehat{\mathcal{X}} = \varprojlim \mathcal{X}_n$ be the inverse limit in the category of topological spaces and continuous maps (cf. [20]). If $\widehat{\pi}_0 : \widehat{\mathcal{X}} \to \mathcal{X}_0$ is the natural continuous map then homeomorphism g of the space $\widehat{\mathcal{X}}$ is said to be a *covering transformation* if the following condition holds

$$\widehat{\pi}_0 = \widehat{\pi}_0 \circ g.$$

The group $\widehat{G}$ of covering homeomorphisms is said to be the *group of covering transformations* of $\mathfrak{S}$. Denote by $G\left(\widehat{\mathcal{X}} \mid \mathcal{X}\right) \stackrel{\text{def}}{=} \widehat{G}$.

Definition 1.28. Let $\mathfrak{S} = \{\mathcal{X}_0 \leftarrow ... \leftarrow \mathcal{X}_n \leftarrow ...\}$ be a finite covering sequence. The pair $\left(\mathcal{Y}, \left\{\pi^{\mathcal{Y}}_n\right\}_{n\in\mathbb{N}}\right)$ of a (discrete) set $\mathcal{Y}$ with and surjective maps $\pi^{\mathcal{Y}}_n : \mathcal{Y} \to \mathcal{X}_n$ is said to be a *coherent system* if for any $n \in \mathbb{N}^0$ a following diagram



is commutative.

Definition 1.29. Let $\mathfrak{S} = \{\mathcal{X}_0 \leftarrow ... \leftarrow \mathcal{X}_n \leftarrow ...\}$ be a topological finite covering sequence. A coherent system $\left(\mathcal{Y}, \left\{\pi^{\mathcal{Y}}_n\right\}\right)$ is said to be a *connected covering* of $\mathfrak{S}$ if $\mathcal{Y}$ is a connected topological space and $\pi^{\mathcal{Y}}_n$ is a regular covering for any $n \in \mathbb{N}$. We will use following notation $\left(\mathcal{Y}, \left\{\pi^{\mathcal{Y}}_n\right\}\right) \downarrow \mathfrak{S}$ or simply $\mathcal{Y} \downarrow \mathfrak{S}$.

Definition 1.30. Let $\left(\mathcal{Y}, \left\{\pi_n^{\mathcal{Y}}\right\}\right)$ be a coherent system of $\mathfrak{S}$ and $y \in \mathcal{Y}$. A subset $\mathcal{V} \subset \mathcal{Y}$ is said to be *special* if $\pi_0^{\mathcal{Y}}\left(\mathcal{V}\right)$ is evenly covered by $\mathcal{X}_1 \to \mathcal{X}_0$ and for any $n \in \mathbb{N}^0$ following conditions hold:

- $\pi_n^{\mathcal{Y}}\left(\mathcal{V}\right) \subset \mathcal{X}_n$ is an open connected set,
- The restriction $\pi_n^{\mathcal{Y}}|_{\mathcal{V}} : \mathcal{V} \to \pi_n^{\mathcal{Y}}\left(\mathcal{V}\right)$ is a bijection.

Remark 1.31. If $\left(\mathcal{Y}, \left\{\pi_n^{\mathcal{Y}}\right\}\right)$ is a covering of $\mathfrak{S}$ then the topology of $\mathcal{Y}$ is generated by special sets.

Definition 1.32. Let us consider the situation of the Definition 1.29. A *morphism* from $\left(\mathcal{Y}', \left\{\pi_n^{\mathcal{Y}'}\right\}\right) \downarrow \mathfrak{S}$ to $\left(\mathcal{Y}'', \left\{\pi_n^{\mathcal{Y}''}\right\}\right) \downarrow \mathfrak{S}$ is a covering $f : \mathcal{Y}' \to \mathcal{Y}''$ such that

$$\pi_n^{\mathcal{Y}''} \circ f = \pi_n^{\mathcal{Y}'}$$

for any $n \in \mathbb{N}$.

1.33. There is a category with objects and morphisms described by Definitions 1.29, 1.32. Denote by $\downarrow \mathfrak{S}$ this category.

Lemma 1.34. *[12] There is the final object of the category $\downarrow \mathfrak{S}$ described in 1.33.*

Definition 1.35. The final object $\left(\widetilde{\mathcal{X}}, \left\{\pi_n^{\widetilde{\mathcal{X}}}\right\}\right)$ of the category $\downarrow \mathfrak{S}$ is said to be the *(topological) inverse limit* of $\downarrow \mathfrak{S}$. The notation $\left(\widetilde{\mathcal{X}}, \left\{\pi_n^{\widetilde{\mathcal{X}}}\right\}\right) = \varprojlim \downarrow \mathfrak{S}$ or simply $\widetilde{\mathcal{X}} = \varprojlim \downarrow \mathfrak{S}$ will be used.

1.4 Hilbert modules

We refer to [3] for definition of Hilbert C^*-modules, or simply Hilbert modules. For any $\xi, \zeta \in X_A$ let us define an A-endomorphism $\theta_{\xi,\zeta}$ given by $\theta_{\xi,\zeta}(\eta) = \xi\langle\zeta,\eta\rangle_{X_A}$ where $\eta \in X_A$. Operator $\theta_{\xi,\zeta}$ shall be denoted by $\xi\rangle\langle\zeta$. Norm completion of algebra generated by operators $\theta_{\xi,\zeta}$ is said to be an algebra of compact operators $\mathcal{K}(X_A)$. We suppose that there is a left action of $\mathcal{K}(X_A)$ on X_A which is A-linear, i.e. action of $\mathcal{K}(X_A)$ commutes with action of A.

1.5 C^*-algebras and von Neumann algebras

In this section I follow to [19].

Definition 1.36. [19] Let $\mathcal{H}$ be a Hilbert space. The *strong* topology on $B\left(\mathcal{H}\right)$ is the locally convex vector space topology associated with the family of seminorms of the form $x \mapsto \|x\xi\|$, $x \in B(\mathcal{H})$, $\xi \in \mathcal{H}$.

Definition 1.37. [19] Let $\mathcal{H}$ be a Hilbert space. The *weak* topology on $B\left(\mathcal{H}\right)$ is the locally convex vector space topology associated with the family of seminorms of the form $x \mapsto |(x\xi,\eta)|$, $x \in B(\mathcal{H})$, $\xi, \eta \in \mathcal{H}$.

Theorem 1.38. *[19] Let M be a C^*-subalgebra of $B(\mathcal{H})$, containing the identity operator. The following conditions are equivalent:*

- *$M = M''$ where M'' is the bicommutant of M;*
- *M is weakly closed;*
- *M is strongly closed.*

Definition 1.39. Any C^*-algebra M is said to be a *von Neumann algebra* or a *W^*- algebra* if M satisfies to the conditions of the Theorem 1.38.

Definition 1.40. [19] Let A be a C^*-algebra, and let S be the state space of A. For any $s \in S$ there is an associated representation $\pi_s : A \to B(\mathcal{H}_s)$. The representation $\bigoplus_{s\in S} \pi_s : A \to \bigoplus_{s\in S} B(\mathcal{H}_s)$ is said to be the *universal representation*. The universal representation can be regarded as $A \to B\left(\bigoplus_{s\in S} \mathcal{H}_s\right)$.

Definition 1.41. [19] Let A be a C^*-algebra, and let $A \to B(\mathcal{H})$ be the universal representation $A \to B(\mathcal{H})$. The strong closure of $\pi(A)$ is said to be the *enveloping von Neumann algebra* or the *enveloping W^*-algebra* of A. The enveloping von Neumann algebra will be denoted by A''.

Theorem 1.42. *[19] For each non-degenerate representation $\pi : A \to B(\mathcal{H})$ of a C^*-algebra A there is a unique normal morphism of A'' onto $\pi(A)''$ which extends π.*

Lemma 1.43. *[19] Let Λ be an increasing net in the partial ordering. Let $\{x_\lambda\}_{\lambda\in\Lambda}$ be an increasing net of self-adjoint operators in $B(\mathcal{H})$, i.e. $\lambda \le \mu$ implies $x_\lambda \le x_\mu$. If $\|x_\lambda\| \le \gamma$ for some $\gamma \in \mathbb{R}$ and all λ then $\{x_\lambda\}$ is strongly convergent to a self-adjoint element $x \in B(\mathcal{H})$ with $\|x_\lambda\| \le \gamma$.*

For each $x \in B(\mathcal{H})$ we define the *range projection* of x (denoted by $[x]$) as projection on the closure of $x\mathcal{H}$. If M is a von Neumann algebra and $x \in M$ then $[x] \in M$.

Proposition 1.44. *[19] For each element x in a von Neumann algebra M there is a unique partial isometry $u \in M$ and positive $|x| \in M_+$ with $uu^* = [|x|]$ and $x = |x|u$.*

Definition 1.45. The formula $x = |x|u$ in the Proposition 1.44 is said to be the *polar decomposition*.

1.46. Any separable C^*-algebra A has a state τ which induces a faithful GNS representation [17]. There is a $\mathbb{C}$-valued product on A given by

$$(a, b) = \tau(a^* b).$$

This product induces a product on $A/\mathcal{I}_\tau$ where $\mathcal{I}_\tau = \{a \in A \mid \tau(a^*a) = 0\}$. So $A/\mathcal{I}_\tau$ is a pre-Hilbert space. Let denote by $L^2(A, \tau)$ the Hilbert completion of $A/\mathcal{I}_\tau$. The Hilbert space $L^2(A, \tau)$ is a space of a GNS representation of A.

1.6 Connections

Definition 1.47. [5] Let $\mathcal{A} \xrightarrow{\rho} \Omega$ be a cycle over $\mathcal{A}$, and $\mathcal{E}$ a finite projective module over $\mathcal{A}$. Then a *connection* ∇ on $\mathcal{E}$ is a linear map $\nabla : \mathcal{E} \to \mathcal{E} \otimes_{\mathcal{A}} \Omega^1$ such that

$$\nabla (\xi x) = \nabla (\xi) x = \xi \otimes d\rho (x) ; \; \forall \xi \in \mathcal{E}, \; \forall x \in \mathcal{A}. \tag{1.2}$$

Here $\mathcal{E}$ is a right module over $\mathcal{A}$ and Ω^1 is considered as a bimodule over $\mathcal{A}$.

Remark 1.48. The definition of the cycle is given in [5].

Proposition 1.49. *[5] Following conditions hold:*

(a) Let $e \in \mathrm{End}_{\mathcal{A}} (\mathcal{E})$ be an idempotent and ∇ is a connection on $\mathcal{E}$; then

$$\xi \mapsto (e \otimes 1) \nabla \xi$$

is a connection on $e\mathcal{E}$,

(b) Any finite projective module $\mathcal{E}$ admits a connection,

(c) The space of connections is an affine space over the vector space $\mathrm{Hom}_{\mathcal{A}} \left(\mathcal{E}, \mathcal{E} \otimes_{\mathcal{A}} \Omega^1\right)$.

1.7 Finite Galois coverings

Here I follow to [1]. Let $A \hookrightarrow \widetilde{A}$ be an injective homomorphism of unital algebras, such that

- $\widetilde{A}$ is a projective finitely generated A-module,
- There is an action $G \times \widetilde{A} \to \widetilde{A}$ of a finite group G such that

$$A = \widetilde{A}^G = \left\{ \widetilde{a} \in \widetilde{A} \mid g\widetilde{a} = \widetilde{a}; \; \forall g \in G \right\}.$$

Let us consider the category $\mathscr{M}^G_{\widetilde{A}}$ of $G - \widetilde{A}$ modules, i.e. any object $M \in \mathscr{M}^G_{\widetilde{A}}$ is a $\widetilde{A}$-module with equivariant action of G, i.e. for any $m \in M$ a following condition holds

$$g (\widetilde{a}m) = (g\widetilde{a}) (gm) \text{ for any } \widetilde{a} \in \widetilde{A}, \; g \in G.$$

Any morphism $\varphi : M \to N$ in the category $\mathscr{M}^G_{\widetilde{A}}$ is G- equivariant, i.e.

$$\varphi (gm) = g\varphi (m) \text{ for any } m \in M, \; g \in G.$$

Let $\widetilde{A} [G]$ be an algebra such that $\widetilde{A} [G] \approx \widetilde{A} \times G$ as an Abelian group and a multiplication law is given by

$$(a, g) (b, h) = (a (gb), gh).$$

The category $\mathscr{M}^G_{\widetilde{A}}$ is equivalent to the category $\mathscr{M}_{\widetilde{A}[G]}$ of $\widetilde{A} [G]$ modules. Otherwise in [1] it is proven that if $\widetilde{A}$ is a finitely generated, projective A-module then there is an equivalence between a category $\mathscr{M}_A$ of A-modules and the category $\mathscr{M}_{\widetilde{A}[G]}$. It turns out that the category $\mathscr{M}^G_{\widetilde{A}}$ is equivalent to the category $\mathscr{M}_A$.

1.8 Spectral triples

This section contains citations of [11].

1.8.1 Definition of spectral triples

Definition 1.50. [11] A (unital) *spectral triple* $(\mathcal{A}, \mathcal{H}, D)$ consists of:

- a pre-C^*-algebra $\mathcal{A}$ with an involution $a \mapsto a^*$, equipped with a faithful representation on:
- a *Hilbert space* $\mathcal{H}$; and also
- a *selfadjoint operator* D on $\mathcal{H}$, with dense domain $\operatorname{Dom} D \subset \mathcal{H}$, such that $a(\operatorname{Dom} D) \subseteq \operatorname{Dom} D$ for all $a \in \mathcal{A}$.

There is a set of axioms for spectral triples described in [11,21]. In this article the regularity axiom is used only.

Axiom 1.51. [21](Regularity) For any $a \in \mathcal{A}$, $[D, a]$ is a bounded operator on $\mathcal{H}$, and both a and $[D, a]$ belong to the domain of smoothness $\bigcap_{k=1}^{\infty} \operatorname{Dom}(\delta^k)$ of the derivation δ on $B(\mathcal{H})$ given by $\delta(T) \stackrel{\text{def}}{=} [|D|, T]$.

Lemma 1.52. *[11] Let $\mathcal{A}$ be an unital Fréchet pre-C^*-algebra, whose C^*-completion is A. If $\tilde{q} = \tilde{q}^2 = \tilde{q}^*$ is a projection in A, then for any $\varepsilon > 0$, we can find a projection $q = q^2 = q^* \in \mathcal{A}$ such that $\|q - \tilde{q}\| < \varepsilon$.*

1.8.2 Representations of smooth algebras

Similarly to [15] we define a representation of $\pi^1 : \mathcal{A} \to B(\mathcal{H}^2)$ given by

$$\pi^1(a) = \begin{pmatrix} a & 0 \\ [D, a] & a \end{pmatrix}. \tag{1.3}$$

We can inductively construct representations $\pi^s : \mathcal{A} \to B\left(\mathcal{H}^{2^s}\right)$ for any $s \in \mathbb{N}$. If π^s is already constructed then $\pi^{s+1} : \mathcal{A} \to B\left(\mathcal{H}^{2^{s+1}}\right)$ is given by

$$\pi^{s+1}(a) = \begin{pmatrix} \pi^s(a) & 0 \\ [D, \pi^s(a)] & \pi^s(a) \end{pmatrix} \tag{1.4}$$

where we assume diagonal action of D on $\mathcal{H}^{2^s}$, i.e.

$$D \begin{pmatrix} x_1 \\ ... \\ x_{2^s} \end{pmatrix} = \begin{pmatrix} Dx_1 \\ ... \\ Dx_{2^s} \end{pmatrix}; \; x_1, ..., x_{2^s} \in \mathcal{H}.$$

For any $s \in \mathbb{N}^0$ there is a seminorm $\|\cdot\|_s$ on $\mathcal{A}$ given by

$$\|a\|_s = \|\pi^s(a)\|. \tag{1.5}$$

The definition of spectral triple requires that $\mathcal{A}$ is a Fréchet algebra with respect to seminorms $\|\cdot\|_s$.

1.8.3 Noncommutative differential forms

Any spectral triple naturally defines a cycle $\rho : \mathcal{A} \to \Omega_D$ (cf. Definition 1.47). In particular for any spectral triple there is an $\mathcal{A}$ module $\Omega^1_D \subset B(\mathcal{H})$ of differential forms which is a linear span of operators given by

$$a\,[D,b]\,;\; a, b \in \mathcal{A}. \tag{1.6}$$

There is differential map

$$\begin{aligned} d : \mathcal{A} &\to \Omega^1_D, \\ a &\mapsto [D,a]\,. \end{aligned} \tag{1.7}$$

2 Noncommutative finite-fold coverings

2.1 Basic construction

Definition 2.1. If A is a C^*- algebra then an action of a group G is said to be *involutive* if $ga^* = (ga)^*$ for any $a \in A$ and $g \in G$. The action is said to be *non-degenerated* if for any nontrivial $g \in G$ there is $a \in A$ such that $ga \neq a$.

Definition 2.2. Let $A \hookrightarrow \widetilde{A}$ be an injective *-homomorphism of unital C^*-algebras. Suppose that there is a non-degenerated involutive action $G \times \widetilde{A} \to \widetilde{A}$ of a finite group G, such that $A = \widetilde{A}^G \stackrel{\text{def}}{=} \left\{ a \in \widetilde{A} \mid a = ga;\ \forall g \in G \right\}$. There is an A-valued product on $\widetilde{A}$ given by

$$\langle a, b \rangle_{\widetilde{A}} = \sum_{g \in G} g\left(a^* b\right) \tag{2.1}$$

and $\widetilde{A}$ is an A-Hilbert module. We say that a triple $\left(A, \widetilde{A}, G\right)$ is an *unital noncommutative finite-fold covering* if $\widetilde{A}$ is a finitely generated projective A-Hilbert module.

Remark 2.3. Above definition is motivated by the Theorem 4.1.

Definition 2.4. Let A, $\widetilde{A}$ be C^*-algebras such that following conditions hold:

(a) There are unital C^*-algebras B, $\widetilde{B}$ and inclusions $A \subset B$, $\widetilde{A} \subset \widetilde{B}$ such that A (resp. B) is an essential ideal of $\widetilde{A}$ (resp. $\widetilde{B}$),

(b) There is an unital noncommutative finite-fold covering $\left(B, \widetilde{B}, G\right)$,

(c)

$$\widetilde{A} = \left\{a \in \widetilde{B} \mid \left\langle \widetilde{B}, a \right\rangle_{\widetilde{B}} \in A\right\}. \tag{2.2}$$

The triple $\left(A, \widetilde{A}, G\right)$ is said to be a *noncommutative finite-fold covering*. The group G is said to be the *covering transformation group* (of $\left(A, \widetilde{A}, G\right)$) and we use the following notation

$$G\left(\widetilde{A} \mid A\right) \stackrel{\text{def}}{=} G. \tag{2.3}$$

Lemma 2.5. *Let us consider the situation of the Definition 2.4. Following conditions hold:*

(i) From (2.2) *it turns out that $\widetilde{A}$ is a closed two sided ideal of $\widetilde{B}$,*

(ii) The action of G on $\widetilde{B}$ is such that $G\widetilde{A} = \widetilde{A}$, i.e. there is the natural action of G on $\widetilde{A}$,

(iii)

$$A \cong \widetilde{A}^G = \left\{a \in \widetilde{A} \mid a = ga;\ \forall g \in G\right\}. \tag{2.4}$$

Remark 2.6. The Definition 2.5 is motivated by the Theorem 4.2.

Definition 2.7. The injective *-homomorphism $A \hookrightarrow \widetilde{A}$, which follows from (2.4) is said to be a *noncommutative finite-fold covering*.

Definition 2.8. Let $\left(A, \widetilde{A}, G\right)$ be a noncommutative finite-fold covering. The algebra $\widetilde{A}$ is a Hilbert A-module with an A-valued product given by

$$\langle a, b \rangle_{\widetilde{A}} = \sum_{g \in G} g(a^* b);\ a, b \in \widetilde{A}. \tag{2.5}$$

We say that this structure of Hilbert A-module is *induced by the covering* $\left(A, \widetilde{A}, G\right)$. Henceforth we shall consider $\widetilde{A}$ as a right A-module, so we will write $\widetilde{A}_A$.

2.2 Induced representation

2.9. Let $\left(A, \widetilde{A}, G\right)$ be a noncommutative finite-fold covering, and let $\rho : A \to B(\mathcal{H})$ be a representation. If $X = \widetilde{A} \otimes_A \mathcal{H}$ is the algebraic tensor product then there is a sesquilinear $\mathbb{C}$-valued product $(\cdot, \cdot)_X$ on X given by

$$(a \otimes \xi, b \otimes \eta)_X = \left(\xi, \langle a, b \rangle_{\widetilde{A}} \eta\right)_{\mathcal{H}} \tag{2.6}$$

where $(\cdot, \cdot)_{\mathcal{H}}$ means the Hilbert space product on $\mathcal{H}$, and $\langle \cdot, \cdot \rangle_{\widetilde{A}}$ is given by (2.5). So X is a pre-Hilbert space. There is a natural map $p : \widetilde{A} \times \left(\widetilde{A} \otimes_A \mathcal{H}\right) \to \widetilde{A} \otimes_A \mathcal{H}$ given by

$$(a, b \otimes \xi) \mapsto ab \otimes \xi.$$

Definition 2.10. Use notation of the Definition 2.8, and 2.9. If $\widetilde{\mathcal{H}}$ is the Hilbert completion of $X = \widetilde{A} \otimes_A \mathcal{H}$ then the map $p : \widetilde{A} \times \left(\widetilde{A} \otimes_A \mathcal{H}\right) \to \widetilde{A} \otimes_A \mathcal{H}$ induces the representation $\widetilde{\rho} : \widetilde{A} \to B\left(\widetilde{\mathcal{H}}\right)$. We say that $\widetilde{\rho}$ *is induced by the pair* $\left(\rho, \left(A, \widetilde{A}, G\right)\right)$.

Remark 2.11. Below any $\widetilde{a} \otimes \xi \in \widetilde{A} \otimes_A \mathcal{H}$ will be regarded as element in $\widetilde{\mathcal{H}}$.

Lemma 2.12. *[12] If $A \to B(\mathcal{H})$ is faithful then $\widetilde{\rho} : \widetilde{A} \to B\left(\widetilde{\mathcal{H}}\right)$ is faithful.*

2.13. Let $\left(A, \widetilde{A}, G\right)$ be a noncommutative finite-fold covering, let $\rho : A \to B(\mathcal{H})$ a faithful representation, and let $\widetilde{\rho} : \widetilde{A} \to B\left(\widetilde{\mathcal{H}}\right)$ is induced by the pair $\left(\rho, \left(A, \widetilde{A}, G\right)\right)$. There is the natural action of G on $\widetilde{\mathcal{H}}$ induced by the map

$$g\left(\widetilde{a} \otimes \xi\right) = \left(g\widetilde{a}\right) \otimes \xi;\ \widetilde{a} \in \widetilde{A},\ g \in G,\ \xi \in \mathcal{H}.$$

There is the natural orthogonal inclusion $\mathcal{H} \subset \widetilde{\mathcal{H}}$ induced by inclusions

$$A \subset \widetilde{A};\ \ A \otimes_A \mathcal{H} \subset \widetilde{A} \otimes_A \mathcal{H}.$$

Action of g on $\widetilde{A}$ can be defined by representation as $g\widetilde{a} = g\widetilde{a}g^{-1}$, i.e.

$$(g\widetilde{a})\xi = g\left(\widetilde{a}\left(g^{-1}\xi\right)\right);\ \forall \xi \in \widetilde{\mathcal{H}}.$$

2.3 Coverings of spectral triples

Definition 2.14. Let $(\mathcal{A}, \mathcal{H}, D)$ be a spectral triple, and let A is the C^*-norm completion of $\mathcal{A}$. Let $\left(A, \widetilde{A}, G\right)$ be an unital noncommutative finite-fold covering. Let $\rho : A \to B(\mathcal{H})$ be a natural representation, and let $\widetilde{\rho} : \widetilde{A} \to B\left(\widetilde{\mathcal{H}}\right)$ be a representation induced by the pair $\left(\rho, \left(A, \widetilde{A}, G\right)\right)$. A spectral triple $\left(\widetilde{\mathcal{A}}, \widetilde{\mathcal{H}}, \widetilde{D}\right)$ is said to be a $\left(A, \widetilde{A}, G\right)$*-lift* of $(\mathcal{A}, \mathcal{H}, D)$ if following conditions hold:

(a) $\widetilde{A}$ is a C^*-norm completion of $\widetilde{\mathcal{A}}$,

(b) $\widetilde{D}\left(1_{\widetilde{A}} \otimes_A \xi\right) = 1_{\widetilde{A}} \otimes_A D\xi;\ \forall \xi \in \mathrm{Dom}\, D$,

(c) $\widetilde{D}\left(g\widetilde{\xi}\right) = g\left(\widetilde{D}\widetilde{\xi}\right)$ for any $\widetilde{\xi} \in \mathrm{Dom}\, \widetilde{D},\ g \in G$.

2.15. Consider the situation of the Definition 2.14. The algebra $\widetilde{A}$ is a finitely generated projective A-module, it turns out following direct sum

$$\widetilde{A} \bigoplus A' = A^n.$$

Let us define the action of G on A^n such that

$$g\overline{a} = g\left(\widetilde{a} + a'\right) = g\widetilde{a} + a'$$

where $\overline{a} = \widetilde{a} + a' \in A^n$, $\widetilde{a} \in \widetilde{A}$ and $a' \in A'$. The action of G on A^n naturally induces an action of G on $\mathrm{End}_A A^n$ given by

$$(g\varphi)(\widetilde{a}) = g \circ \varphi\left(g^{-1}\widetilde{a}\right); \text{ where } g \in G,\ \widetilde{a} \in A^n,\ \varphi \in \mathrm{End}_A A^n.$$

There is the natural bijection $\mathbb{M}_n(A) \approx \mathrm{End}_A A^n$, so there is a natural action of G on $\mathbb{M}_n(A)$. There is a projection $p \in \mathbb{M}_n(A)$ such that

$$\widetilde{A} = pA^n.$$

From the definition of the action of G on $\mathbb{M}_n(A)$ it follows that $g \circ p = p$ for any $g \in G$. The subalgebra $\mathbb{M}_n(\mathcal{A}) \subset \mathbb{M}_n(A)$ is dense, it turns out that for any $\varepsilon > 0$ there is an idempotent $f \in \mathbb{M}_n(\mathcal{A})$ such that $\|p - f\| < \varepsilon$. If

$$f_{\mathrm{inv}} = \frac{\sum_{g\in G} gf}{|G|}$$

then following conditions hold:

$$\|p - f\| < \varepsilon,$$
$$gf_{\mathrm{inv}} = f_{\mathrm{inv}}; \text{ for any } g \in G.$$

From the Lemma 1.52 it follows that there is a projection $\widetilde{p} \in \mathbb{M}_n(A)$ which is similar to f_{inv} and p such that following conditions hold:

$$g\widetilde{p} = \widetilde{p}; \text{ for any } g \in G,$$
$$\widetilde{p} \in \mathbb{M}_n(\mathcal{A}),$$
$$\widetilde{p}A^n \approx \widetilde{A}.$$

Let $\widetilde{\mathcal{E}} = \widetilde{p}\mathcal{A}^n$ be a projective $\mathcal{A}$-module, let Ω^1_D be given by (1.6). From the Proposition 1.49 it follows that there is a connection

$$\nabla' : \widetilde{\mathcal{E}} \to \widetilde{\mathcal{E}} \otimes_{\mathcal{A}} \Omega^1_D.$$

Let us define a connection

$$\widetilde{\nabla} : \widetilde{\mathcal{E}} \to \widetilde{\mathcal{E}} \otimes_{\mathcal{A}} \Omega^1_D,$$
$$\widetilde{\nabla}(\widetilde{a}) = \frac{1}{|G|} \sum_{g\in G} g^{-1}\left(\nabla'(g\widetilde{a})\right)$$

The connection $\widetilde{\nabla}$ is equivariant, i.e.

$$\widetilde{\nabla}(g\widetilde{a}) = g\left(\widetilde{\nabla}(\widetilde{a})\right); \text{ for any } g \in G,\ \widetilde{a} \in \widetilde{\mathcal{E}}. \tag{2.7}$$

Let $\mathcal{H}^\infty = \bigcap_{n=1}^\infty \operatorname{Dom} D^n$, and let us define an operator $\widetilde{D} : \widetilde{\mathcal{E}} \otimes_{\mathcal{A}} \mathcal{H}^\infty \to \widetilde{\mathcal{E}} \otimes_{\mathcal{A}} \mathcal{H}^\infty$ such that if $\widetilde{a} \in \widetilde{\mathcal{E}}$ and

$$\nabla\left(\widetilde{a}\right) = \sum_{j=1}^m \widetilde{a}_j \otimes \omega_j$$

then

$$\widetilde{D}\left(\widetilde{a} \otimes \xi\right) = \sum_{j=1}^m \widetilde{a}_j \otimes \omega_j\left(\xi\right) + \widetilde{a} \otimes D\xi. \tag{2.8}$$

The space $\widetilde{\mathcal{E}} \otimes_{\mathcal{A}} \mathcal{H}^\infty$ is a dense subspace of the Hilbert space $\widetilde{\mathcal{H}} = \widetilde{A} \otimes_{A} \mathcal{H}$. It turns out $\widetilde{D}$ can be regarded as an unbounded operator on $\widetilde{\mathcal{H}}$. Denote by $\widetilde{\delta}$ a derivation on $B\left(\widetilde{\mathcal{H}}\right)$ given by

$$\widetilde{\delta}(\widetilde{T}) \stackrel{\text{def}}{=} \left[\left|\widetilde{D}\right|, \widetilde{T}\right].$$

Denote by

$$\widetilde{\mathcal{A}} = \left\{\widetilde{a} \in \widetilde{A} = \widetilde{p}A^n \mid \left[\widetilde{D}, \widetilde{a}\right] \in B\left(\widetilde{\mathcal{H}}\right) \ \&\ \widetilde{a}, \left[\widetilde{D}, \widetilde{a}\right] \in \bigcap_{k=1}^\infty \operatorname{Dom}\left(\widetilde{\delta}^k\right)\right\}. \tag{2.9}$$

From the above equation it turns out:

- $\widetilde{\mathcal{A}}$ is a subalgebra of $\widetilde{A}$,
- $G\widetilde{\mathcal{A}} = \widetilde{\mathcal{A}}$,
- $\widetilde{\mathcal{A}} = \widetilde{\mathcal{E}}$.

Using $\widetilde{\mathcal{A}} = \widetilde{\mathcal{E}}$ on can write

$$\widetilde{\nabla} : \widetilde{\mathcal{A}} \to \widetilde{\mathcal{A}} \otimes_{\mathcal{A}} \Omega_D^1$$

instead of

$$\widetilde{\nabla} : \widetilde{\mathcal{E}} \to \widetilde{\mathcal{E}} \otimes_{\mathcal{A}} \Omega_D^1.$$

Lemma 2.16. *In the above situation there is the unique G-equivariant connection*

$$\widetilde{\nabla} : \widetilde{\mathcal{A}} \to \widetilde{\mathcal{A}} \otimes_{\mathcal{A}} \Omega_D^1.$$

Proof. It follows from the Proposition 1.49 that the space of connections is an affine space over the vector space $\operatorname{Hom}_{\mathcal{A}}\left(\widetilde{\mathcal{A}}, \widetilde{\mathcal{A}} \otimes_{\mathcal{A}} \Omega_D^1\right)$. The space of G-equivariant connections is an affine space over the vector space $\operatorname{Hom}_{\mathcal{A}}^G\left(\widetilde{\mathcal{A}}, \widetilde{\mathcal{A}} \otimes_{\mathcal{A}} \Omega_D^1\right)$ of G-equivariant morphisms, i.e. morphisms in the category $\mathscr{M}_{\widetilde{\mathcal{A}}}^G$ (cf. 1.7). However from 1.7 it follows that the category $\mathscr{M}_{\widetilde{\mathcal{A}}}^G$ is equivalent to the category $\mathscr{M}_{\mathcal{A}}$ of $\mathcal{A}$-modules. It turns out that there is a 1-1 correspondence between connections

$$\nabla : \mathcal{A} \to \mathcal{A} \otimes_{\mathcal{A}} \Omega_D^1 = \Omega_D^1$$

and G-equivariant connections

$$\widetilde{\nabla} : \widetilde{\mathcal{A}} \to \widetilde{\mathcal{A}} \otimes_A \Omega^1_D.$$

It follows that thee is the unique G-equivariant $\widetilde{\nabla}$ connection which corresponds to

$$\nabla : \mathcal{A} \to \mathcal{A} \otimes_{\mathcal{A}} \Omega^1_D = \Omega^1_D,$$
$$a \mapsto [D, a].$$

□

Above reasonings give a following theorem.

Theorem 2.17. *If $\widetilde{A}$ is given by* (2.9), $\widetilde{\mathcal{H}} = \widetilde{A} \otimes_A \mathcal{H}$ *and $\widetilde{D}$ is given by* (2.8), *then a spectral triple $\left(\widetilde{\mathcal{A}}, \widetilde{\mathcal{H}}, \widetilde{D}\right)$ is a $\left(A, \widetilde{A}, G\right)$-lift of $(\mathcal{A}, \mathcal{H}, D)$. There is the unique $\left(A, \widetilde{A}, G\right)$-lift of $(\mathcal{A}, \mathcal{H}, D)$.*

Remark 2.18. From (2.7) it follows that the operator $\widetilde{D}$ is equivariant, i.e.

$$\widetilde{D}\left(g\widetilde{\xi}\right) = g\left(\widetilde{D}\widetilde{\xi}\right);\ \forall\widetilde{\xi} \in \mathrm{Dom}\,\widetilde{G},\ \forall g \in G. \tag{2.10}$$

There are two equivalent ways of definition of operator $\widetilde{D}$ from the Theorem 2.17:

(a) Looking for an operator $\widetilde{D}$ which satisfies to the Definition 2.14,

(b) Application of the equation (2.8).

3 Noncommutative infinite coverings

3.1 Basic construction

This section contains a noncommutative generalization of infinite coverings.

Definition 3.1. Let

$$\mathfrak{S} = \left\{A = A_0 \xrightarrow{\pi_1} A_1 \xrightarrow{\pi_2} ... \xrightarrow{\pi_n} A_n \xrightarrow{\pi^{n+1}} ...\right\}$$

be a sequence of C^*-algebras and noncommutative finite-fold coverings such that:

(a) Any composition $\pi_{n_1} \circ ... \circ \pi_{n_0+1} \circ \pi_{n_0} : A_{n_0} \to A_{n_1}$ corresponds to the noncommutative covering $(A_{n_0}, A_{n_1}, G(A_{n_1} \mid A_{n_0}))$;

(b) If $k < l < m$ then $G(A_m \mid A_k) A_l = A_l$ (Action of $G(A_m \mid A_k)$ on A_l means that $G(A_m \mid A_k)$ acts on A_m, so $G(A_m \mid A_k)$ acts on A_l since A_l a subalgebra of A_m);

(c) If $k < l < m$ are nonegative integers then there is the natural exact sequence of covering transformation groups

$$\{e\} \to G(A_m \mid A_l) \xrightarrow{\iota} G(A_m \mid A_k) \xrightarrow{\pi} G(A_l \mid A_k) \to \{e\}$$

where the existence of the homomorphism $G(A_m \mid A_k) \xrightarrow{\pi} G(A_l \mid A_k)$ follows from (b).

The sequence $\mathfrak{S}$ is said to be an *(algebraical) finite covering sequence*. For any finite covering sequence we will use the notation $\mathfrak{S} \in \mathfrak{FinAlg}$.

Definition 3.2. Let $\widehat{A} = \varinjlim A_n$ be the C^*-inductive limit [17], and suppose that $\widehat{G} = \varprojlim G(A_n \mid A)$ is the projective limit of groups [20]. There is the natural action of $\widehat{G}$ on $\widehat{A}$. A non-degenerate faithful representation $\widehat{A} \to B(\mathcal{H})$ is said to be *equivariant* if there is an action of $\widehat{G}$ on $\mathcal{H}$ such that for any $\xi \in \mathcal{H}$ and $g \in \widehat{G}$ the following condition holds

$$(ga)\,\xi = g\left(a\left(g^{-1}\xi\right)\right). \tag{3.1}$$

Example 3.3. Let S be the state space of $\widehat{A}$, and let $\widehat{A} \to B\left(\bigoplus_{s\in S} \mathcal{H}_s\right)$ be the universal representation. There is the natural action of $\widehat{G}$ on S given by

$$(gs)\,(a) = s\,(ga)\,;\; s \in S,\; a \in \widehat{A},\; g \in \widehat{G}.$$

The action of $\widehat{G}$ on S induces the action of $\widehat{G}$ on $\bigoplus_{s\in S} \mathcal{H}_s$. It follows that the universal representation is equivariant.

Example 3.4. Let s be a faithful state which corresponds to the representation $\widehat{A} \to B(\mathcal{H}_s)$ and $\left\{g_n \in \widehat{G}\right\}_{n\in\mathbb{N}} = \widehat{G}$ is a bijection. The state

$$\sum_{n\in\mathbb{N}} \frac{g_n s}{2^n}$$

corresponds to an equvariant representation $\widehat{A} \to B\left(\bigoplus_{g\in\widehat{G}} \mathcal{H}_{gs}\right)$.

Definition 3.5. Let $\pi : \widehat{A} \to B(\mathcal{H})$ be an equivariant representation. A positive element $\overline{a} \in B(\mathcal{H})_+$ is said to be *special* (with respect to π) if following conditions hold:

(a) For any $n \in \mathbb{N}^0$ the following series

$$a_n = \sum_{g\in\ker\left(\widehat{G}\to G(A_n \mid A)\right)} g\overline{a}$$

is strongly convergent and the sum lies in A_n, i.e. $a_n \in A_n$;

(b) If $f_\varepsilon : \mathbb{R} \to \mathbb{R}$ is given by

$$f_\varepsilon(x) = \begin{cases} 0 & x \le \varepsilon \\ x - \varepsilon & x > \varepsilon \end{cases} \tag{3.2}$$

then for any $n \in \mathbb{N}^0$ and for any $z \in A$ following series

$$b_n = \sum_{g \in \ker\left(\widehat{G} \to G(A_n \mid A)\right)} g\left(z\overline{a}z^*\right),$$

$$c_n = \sum_{g \in \ker\left(\widehat{G} \to G(A_n \mid A)\right)} g\left(z\overline{a}z^*\right)^2,$$

$$d_n = \sum_{g \in \ker\left(\widehat{G} \to G(A_n \mid A)\right)} g f_\varepsilon\left(z\overline{a}z^*\right)$$

are strongly convergent and the sums lie in A_n, i.e. $b_n,\ c_n,\ d_n \in A_n$;

(c) For any $\varepsilon > 0$ there is $N \in \mathbb{N}$ (which depends on $\overline{a}$ and z) such that for any $n \geq N$ a following condition holds

$$\left\| b_n^2 - c_n \right\| < \varepsilon. \tag{3.3}$$

An element $\overline{a}' \in B(\mathcal{H})$ is said to be *weakly special* if

$$\overline{a}' = x\overline{a}y; \text{ where } x, y \in \widehat{A}, \text{ and } \overline{a} \in B(\mathcal{H}) \text{ is special.}$$

Lemma 3.6. *[12] If $\overline{a} \in B(\mathcal{H})_+$ is a special element and $G_n = \ker\left(\widehat{G} \to G(A_n \mid A)\right)$ then from*

$$a_n = \sum_{g \in G_n} g\overline{a},$$

it follows that $\overline{a} = \lim_{n\to\infty} a_n$ in the sense of the strong convergence. Moreover one has $\overline{a} = \inf_{n\in\mathbb{N}} a_n$.

Corollary 3.7. *[12] Any weakly special element lies in the enveloping von Neumann algebra $\widehat{A}''$ of $\widehat{A} = \varinjlim A_n$. If $\overline{A}_\pi \subset B(\mathcal{H})$ is the C^*-norm completion of an algebra generated by weakly special elements then $\overline{A}_\pi \subset \widehat{A}''$.*

Lemma 3.8. *[12] If $\overline{a} \in B(\mathcal{H})$ is special, (resp. $\overline{a}' \in B(\mathcal{H})$ weakly special) then for any $g \in \widehat{G}$ the element $g\overline{a}$ is special, (resp. $g\overline{a}'$ is weakly special).*

Corollary 3.9. *[12] If $\overline{A}_\pi \subset B(\mathcal{H})$ is the C^*-norm completion of algebra generated by weakly special elements, then there is a natural action of $\widehat{G}$ on $\overline{A}_\pi$.*

Definition 3.10. Let $\mathfrak{S} = \left\{ A = A_0 \xrightarrow{\pi^1} A_1 \xrightarrow{\pi^2} \ldots \xrightarrow{\pi^n} A_n \xrightarrow{\pi^{n+1}} \ldots \right\}$ be an algebraical finite covering sequence. Let $\pi : \widehat{A} \to B(\mathcal{H})$ be an equivariant representation. Let $\overline{A}_\pi \subset B(\mathcal{H})$ be the C^*-norm completion of algebra generated by weakly special elements. We say that $\overline{A}_\pi$ is the *disconnected inverse noncommutative limit* of $\downarrow \mathfrak{S}$ (*with respect to* π). The triple $\left(A, \overline{A}_\pi, G\left(\overline{A}_\pi \mid A\right) \overset{\text{def}}{=} \widehat{G}\right)$ is said to be the *disconnected infinite noncommutative covering* of $\mathfrak{S}$ (*with respect to* π). If π is the universal representation then "with respect to π" is dropped and we will write $\left(A, \overline{A}, G\left(\overline{A} \mid A\right)\right)$.

Definition 3.11. Any maximal irreducible subalgebra $\widetilde{A}_\pi \subset \overline{A}_\pi$ is said to be a *connected component* of $\mathfrak{S}$ (*with respect to* π). The maximal subgroup $G_\pi \subset G\left(\overline{A}_\pi \mid A\right)$ among subgroups $G \subset G\left(\overline{A}_\pi \mid A\right)$ such that $G\widetilde{A}_\pi = \widetilde{A}_\pi$ is said to be the $\widetilde{A}_\pi$*-invariant group* of $\mathfrak{S}$. If π is the universal representation then "with respect to π" is dropped.

Remark 3.12. From the Definition 3.11 it follows that $G_\pi \subset G\left(\overline{A}_\pi \mid A\right)$ is a normal subgroup.

Definition 3.13. Let

$$\mathfrak{S} = \left\{A = A_0 \xrightarrow{\pi^1} A_1 \xrightarrow{\pi^2} ... \xrightarrow{\pi^n} A_n \xrightarrow{\pi^{n+1}} ...\right\} \in \mathfrak{FinAlg},$$

and let $\left(A, \overline{A}_\pi, G\left(\overline{A}_\pi \mid A\right)\right)$ be a disconnected infinite noncommutative covering of $\mathfrak{S}$ with respect to an equivariant representation $\pi : \varinjlim A_n \to B\left(\mathcal{H}\right)$. Let $\widetilde{A}_\pi \subset \overline{A}_\pi$ be a connected component of $\mathfrak{S}$ with respect to π, and let $G_\pi \subset G\left(\overline{A}_\pi \mid A\right)$ be the $\widetilde{A}_\pi$ - invariant group of $\mathfrak{S}$. Let $h_n : G\left(\overline{A}_\pi \mid A\right) \to G\left(A_n \mid A\right)$ be the natural surjective homomorphism. The representation $\pi : \varinjlim A_n \to B\left(\mathcal{H}\right)$ is said to be *good* if it satisfies to following conditions:

(a) The natural *-homomorphism $\varinjlim A_n \to M\left(\widetilde{A}_\pi\right)$ is injective,

(b) If $J \subset G\left(\overline{A}_\pi \mid A\right)$ is a set of representatives of $G\left(\overline{A}_\pi \mid A\right)/G_\pi$, then the algebraic direct sum

$$\bigoplus_{g \in J} g\widetilde{A}_\pi$$

is a dense subalgebra of $\overline{A}_\pi$,

(c) For any $n \in \mathbb{N}$ the restriction $h_n|_{G_\pi}$ is an epimorphism, i. e. $h_n\left(G_\pi\right) = G\left(A_n \mid A\right)$.

If π is the universal representation we say that $\mathfrak{S}$ is *good*.

Definition 3.14. Let $\mathfrak{S} = \{A = A_0 \to A_1 \to ... \to A_n \to ...\} \in \mathfrak{FinAlg}$ be an algebraical finite covering sequence. Let $\pi : \widehat{A} \to B\left(\mathcal{H}\right)$ be a good representation. A connected component $\widetilde{A}_\pi \subset \overline{A}_\pi$ is said to be the *inverse noncommutative limit of* $\downarrow \mathfrak{S}$ *(with respect to* π*)*. The $\widetilde{A}_\pi$-invariant group G_π is said to be the *covering transformation group of* $\mathfrak{S}$ (*with respect to* π). The triple $\left(A, \widetilde{A}_\pi, G_\pi\right)$ is said to be the *infinite noncommutative covering* of $\mathfrak{S}$ (*with respect to* π). We will use the following notation

$$\varprojlim_\pi \downarrow \mathfrak{S} \stackrel{\text{def}}{=} \widetilde{A}_\pi,$$

$$G\left(\widetilde{A}_\pi \mid A\right) \stackrel{\text{def}}{=} G_\pi.$$

If π is the universal representation then "with respect to π" is dropped and we will write $\left(A, \widetilde{A}, G\right)$, $\varprojlim \downarrow \mathfrak{S} \stackrel{\text{def}}{=} \widetilde{A}$ and $G\left(\widetilde{A} \mid A\right) \stackrel{\text{def}}{=} G$.

Definition 3.15. Let $\mathfrak{S} = \{A = A_0 \to A_1 \to ... \to A_n \to ...\} \in \mathfrak{FinAlg}$ be an algebraical finite covering sequence. Let $\pi : \widehat{A} \to B(\mathcal{H})$ be a good representation. Let $\left(A, \widetilde{A}_\pi, G_\pi\right)$ be the infinite noncommutative covering of $\mathfrak{S}$ (with respect to π). Let $K\left(\widetilde{A}_\pi\right)$ be the Pedersen ideal of $\widetilde{A}_\pi$. We say that $\mathfrak{S}$ *allows inner product (with respect to π)* if following conditions hold

(a) Any $\widetilde{a} \in K\left(\widetilde{A}_\pi\right)$ is weakly special,

(b) For any $n \in \mathbb{N}$, and $\widetilde{a}, \widetilde{b} \in K\left(\widetilde{A}_\pi\right)$ the series

$$a_n = \sum_{g \in \ker\left(\widehat{G} \to G(A_n \mid A)\right)} g\left(\widetilde{a}^* \widetilde{b}\right)$$

is strongly convergent and $a_n \in A_n$.

Remark 3.16. If $\mathfrak{S}$ allows inner product (with respect to π) then $K\left(\widetilde{A}_\pi\right)$ is a pre-Hilbert A module such that the inner product is given by

$$\left\langle \widetilde{a}, \widetilde{b} \right\rangle = \sum_{g \in \widehat{G}} g\left(\widetilde{a}^* \widetilde{b}\right) \in A$$

where the above series is strongly convergent. The completion of $K\left(\widetilde{A}_\pi\right)$ with respect to a norm

$$\|\widetilde{a}\| = \sqrt{\|\langle \widetilde{a}, \widetilde{a} \rangle\|}$$

is an A-Hilbert module. Denote by X_A this completion. The ideal $K\left(\widetilde{A}_\pi\right)$ is a left $\widetilde{A}_\pi$-module, so X_A is also $\widetilde{A}_\pi$-module. Sometimes we will write ${}_{\widetilde{A}_\pi}X_A$ instead X_A.

Definition 3.17. Let $\mathfrak{S} = \{A = A_0 \to A_1 \to ... \to A_n \to ...\} \in \mathfrak{FinAlg}$ and $\mathfrak{S}$ allows inner product (with respect to π) then $K\left(\widetilde{A}_\pi\right)$ then we say that given by the Remark 3.16 A-Hilbert module ${}_{\widetilde{A}_\pi}X_A$ *corresponds to the pair* $(\mathfrak{S}, \pi)$. If π is the universal representation then we say that ${}_{\widetilde{A}}X_A$ *corresponds to* $\mathfrak{S}$.

3.2 Induced representation

Let $\pi : \widehat{A} \to B\left(\overline{\mathcal{H}}_\pi\right)$ be a good representation. Let $\left(A, \widetilde{A}_\pi, G_\pi\right)$ be an infinite noncommutative covering with respect to π of $\mathfrak{S}$. Denote by $\overline{W}_\pi \subset B\left(\overline{\mathcal{H}}_\pi\right)$ the $\widehat{A}$-bimodule of weakly special elements, and denote by

$$\widetilde{W}_\pi = \overline{W}_\pi \bigcap \widetilde{A}_\pi. \tag{3.4}$$

If π is the universal representation then we write $\widetilde{W}$ instead $\widetilde{W}_\pi$.

Lemma 3.18. *[12] If $\widetilde{a}, \widetilde{b} \in \widetilde{W}_\pi$ are weakly special elements then a series*

$$\sum_{g \in G_\pi} g\left(\widetilde{a}^* \widetilde{b}\right)$$

is strongly convergent.

Definition 3.19. Element $\widetilde{a} \in \widetilde{A}_\pi$ is said to be *square-summable* if the series

$$\sum_{g \in G_\pi} g\left(\widetilde{a}^* \widetilde{a}\right) \tag{3.5}$$

is strongly convergent to a bounded operator. Denote by $L^2\left(\widetilde{A}_\pi\right)$ (or $L^2\left(\widetilde{A}\right)$ the $\mathbb{C}$-space of square-summable operators.

Remark 3.20. If $\widetilde{b} \in \widehat{A}$, and $\widetilde{a} \in L^2\left(\widetilde{A}\right)$ then

$$\left\|\sum_{g \in G_\pi} g\left(\widetilde{b}\widetilde{a}\right)^*\left(\widetilde{b}\widetilde{a}\right)\right\| \le \left\|\widetilde{b}\right\|^2 \left\|\sum_{g \in G_\pi} g\left(\widetilde{a}^*\widetilde{a}\right)\right\|, \quad \left\|\sum_{g \in G_\pi} g\left(\widetilde{a}\widetilde{b}\right)^*\left(\widetilde{a}\widetilde{b}\right)\right\| \le \left\|\widetilde{b}\right\|^2 \left\|\sum_{g \in G_\pi} g\left(\widetilde{a}^*\widetilde{a}\right)\right\|$$

it turns out

$$\widehat{A}L^2\left(\widetilde{A}_\pi\right) \subset L^2\left(\widetilde{A}_\pi\right), \; L^2\left(\widetilde{A}_\pi\right)\widehat{A} \subset L^2\left(\widetilde{A}_\pi\right), \tag{3.6}$$

i.e. there is the left and right action of $\widehat{A}$ on $L^2\left(\widetilde{A}\right)$.

Remark 3.21. If $a, b \in L^2\left(\widetilde{A}_\pi\right)$ then sum $\sum_{g \in G_\pi} g\left(\widetilde{a}^*\widetilde{b}\right) \in \widehat{A}''$ is bounded and G_π-invariant, hence $\sum_{g \in G_\pi} g\left(\widetilde{a}^*\widetilde{b}\right) \in A''$.

Remark 3.22. From the Lemma 3.18 it turns out $\widetilde{W}_\pi \subset L^2\left(\widetilde{A}_\pi\right)$

3.23. Let $A \to B(\mathcal{H})$ be a representation. Denote by $\widetilde{\mathcal{H}}$ a Hilbert completion of a pre-Hilbert space

$$L^2\left(\widetilde{A}_\pi\right) \otimes_A \mathcal{H},$$

$$\text{with a scalar product } \left(\widetilde{a} \otimes \xi, \widetilde{b} \otimes \eta\right)_{\widetilde{\mathcal{H}}} = \left(\xi, \left(\sum_{g \in G_\pi} g\left(\widetilde{a}^*\widetilde{b}\right)\right)\eta\right)_{\mathcal{H}}. \tag{3.7}$$

There is the left action of $\widehat{A}$ on $L^2\left(\widetilde{A}_\pi\right) \otimes_A \mathcal{H}$ given by

$$\widetilde{b}\left(\widetilde{a} \otimes \xi\right) = \widetilde{b}\widetilde{a} \otimes \xi$$

where $\widetilde{a} \in L^2\left(\widetilde{A}_\pi\right)$, $\widetilde{b} \in \widehat{A}$, $\xi \in \mathcal{H}$. The left action of $\widehat{A}$ on $L^2\left(\widetilde{A}_\pi\right) \otimes_A \mathcal{H}$ induces following representations

$$\widehat{\rho} : \widehat{A} \to B\left(\widetilde{\mathcal{H}}\right),$$

$$\widetilde{\rho} : \widetilde{A}_\pi \to B\left(\widetilde{\mathcal{H}}\right).$$

Definition 3.24. The constructed in 3.23 representation $\widetilde{\rho} : \widetilde{A}_\pi \to B\left(\widetilde{\mathcal{H}}\right)$ is said to be *induced* by $(\rho, \mathfrak{S}, \pi)$. We also say that $\widetilde{\rho}$ is *induced* by $\left(\rho, \left(A, \widetilde{A}_\pi, G\left(\widetilde{A}_\pi \mid A\right)\right), \pi\right)$. If π is an universal representation we say that $\widetilde{\rho}$ is *induced* by $(\rho, \mathfrak{S})$ and/or $\left(\rho, \left(A, \widetilde{A}, G\left(\widetilde{A} \mid A\right)\right)\right)$.

Remark 3.25. If ρ is faithful, then ρ is faithful.

Remark 3.26. There is an action of G_π on $\widetilde{\mathcal{H}}$ induced by the natural action of G_π on the $\widetilde{A}_\pi$-bimodule $L^2\left(\widetilde{A}_\pi\right)$. If the representation $\widetilde{A}_\pi \to B\left(\widetilde{\mathcal{H}}\right)$ is faithful then an action of G_π on $\widetilde{A}_\pi$ is given by

$$(g\widetilde{a})\,\xi = g\left(\widetilde{a}\left(g^{-1}\widetilde{\xi}\right)\right);\ \forall g \in G,\ \forall \widetilde{a} \in \widetilde{A}_\pi,\ \forall \widetilde{\xi} \in \widetilde{\mathcal{H}}.$$

3.27. If $\mathfrak{S}$ allows inner product with respect to π then for any representation $A \to B(\mathcal{H})$ an algebraic tensor product ${}_{\widetilde{A}_\pi}X_A \otimes_A \mathcal{H}$ is a pre-Hilbert space with the product given by

$$(a \otimes \xi, b \otimes \eta) = (\xi, \langle a, b\rangle\,\eta)$$

(cf. Definitions 3.17 and 3.15)

Lemma 3.28. *[12] Suppose $\mathfrak{S}$ allows inner product with respect to π and any $\widetilde{a} \in K\left(\widetilde{A}_\pi\right)$ is weakly special. If $\widetilde{\mathcal{H}}$ (resp. $\widetilde{\mathcal{H}}'$) is a Hilbert norm completion of $W_\pi \otimes_A \mathcal{H}$ (resp. ${}_{\widetilde{A}_\pi}X_A \otimes_A \mathcal{H}$) then there is the natural isomorphism $\widetilde{\mathcal{H}} \cong \widetilde{\mathcal{H}}'$.*

3.29. Let $\mathcal{H}_n$ be a Hilbert completion of $A_n \otimes_A \mathcal{H}$ which is constructed in the section 2.2. Clearly

$$L^2\left(\widetilde{A}_\pi\right) \otimes_{A_n} \mathcal{H}_n = L^2\left(\widetilde{A}_\pi\right) \otimes_{A_n} (A_n \otimes_A \mathcal{H}) = L^2\left(\widetilde{A}_\pi\right) \otimes_A \mathcal{H}. \tag{3.8}$$

3.3 Coverings of spectral triples

Definition 3.30. Let $(\mathcal{A}, \mathcal{H}, D)$ be a spectral triple, and let A be the C^*-norm completion of $\mathcal{A}$ with the natural representation $A \to B(\mathcal{H})$. Let

$$\mathfrak{S} = \left\{A = A_0 \xrightarrow{\pi_1} A_1 \xrightarrow{\pi_2} ... \xrightarrow{\pi_n} A_n \xrightarrow{\pi^{n+1}} ...\right\} \in \mathfrak{FinAlg} \tag{3.9}$$

be a good algebraical finite covering sequence. Suppose that for any $n > 0$ there is a spectral triple $(\mathcal{A}_n, \mathcal{H}_n, D_n)$, such that

- A_n is the C^*-norm completion of $\mathcal{A}_n$,
- There is a good representation $\pi : \varinjlim A_n \to B(\mathcal{H}_\pi)$,
- For any $k > l \geq 0$ the spectral triple $(\mathcal{A}_k, \mathcal{H}_k, D_k)$ is a $(A_l, A_k, G(A_k \mid A_l))$-lift of $(\mathcal{A}_l, \mathcal{H}_l, D_l)$.

We say that

$$\mathfrak{S}_{(\mathcal{A},\mathcal{H},D)} = \{(\mathcal{A},\mathcal{H},D) = (\mathcal{A}_0,\mathcal{H}_0,D_0), (\mathcal{A}_1,\mathcal{H}_1,D_1), ..., (\mathcal{A}_n,\mathcal{H}_n,D_n)\} \tag{3.10}$$

is a *coherent sequence of spectral triples*. We write $\mathfrak{S}_{(\mathcal{A},\mathcal{H},D)} \in \mathfrak{CohTriple}$.

Let us consider a coherent sequence of spectral triples. Let $\left(A, \widetilde{A}_\pi, G_\pi\right)$ be an infinite noncommutative covering (with respect to π) of $\mathfrak{S}$. Denote by $L^2\left(\widetilde{A}_\pi\right) \subset \widetilde{A}_\pi$ the space of square-summable elements, and denote by $J_n = \ker\left(G_\pi \to G\left(A_n \mid A\right)\right)$. Let us consider a square-summable element $\widetilde{a} \in L^2\left(\widetilde{A}_\pi\right)$ and denote by

$$a_n = \sum_{g \in J_n} g\widetilde{a} \in A_n.$$

Let $\widetilde{\rho} : \widetilde{A}_\pi \to \widetilde{\mathcal{H}}$ be induced by $\left(\rho, \left(A, \widetilde{A}_\pi, G\left(\widetilde{A}_\pi \mid A\right)\right), \pi\right)$.

Definition 3.31. In the above situation weakly special element $\widetilde{a} \in \widetilde{A}_\pi$ is said to by $\mathfrak{S}_{(\mathcal{A},\mathcal{H},D)}$*-smooth with respect to* π (or $\mathfrak{S}_{(\mathcal{A},\mathcal{H},D)}$*-smooth* if π is the universal representation) if following conditions hold:

(a) $a_n \in \mathcal{A}_n$ for any $n \in \mathbb{N}$.

(b) For any $s \in \mathbb{N}$ the sequence $\left\{1_{M(\widetilde{A})} \otimes \pi_n^s\left(a_n\right) \in B\left(\widetilde{\mathcal{H}}^{2^s}\right)\right\}_{n\in\mathbb{N}}$ is strongly convergent. (The representation $\pi_n^s : \mathcal{A}_n \to B\left(\mathcal{H}_n^{2^s}\right)$ is given by (1.4)).

(c) The sequence $\left\{1_{M(\widetilde{A})} \otimes [D_n, a_n] \in B\left(\widetilde{\mathcal{H}}\right)\right\}_{n\in\mathbb{N}}$ is strongly convergent and

$$\lim_{n\to\infty} 1_{M(\widetilde{A})} \otimes [D_n, a_n] \in L^2\left(\widetilde{A}_\pi\right) \otimes_A \Omega_D^1 \subset B\left(\widetilde{\mathcal{H}}\right), \text{ (cf. Remark 3.32).}$$

(d) The element $\widetilde{a}$ lies in the Pedersen ideal of $\widetilde{A}_\pi$, i.e. $\widetilde{a} \in K\left(\widetilde{A}_\pi\right)$.

Denote by $\widetilde{a}^s = \lim_{n\to\infty} 1_{M(\widetilde{A})} \otimes \pi_n^s\left(a_n\right)$ in sense the strong convergence, and denote by $\widetilde{W}_\pi^\infty$ the space of smooth elements. If π is the universal representation then we write $\widetilde{W}^\infty$ instead $\widetilde{W}_\pi^\infty$.

Remark 3.32. From (3.8) it follows that $1_{M(\widetilde{A})} \otimes \pi_n^s\left(a_n\right)$ (resp. $1_{M(\widetilde{A})} \otimes_{A_n} [D_n, a_n]$) can be regarded as an operator in $B\left(\widetilde{\mathcal{H}}^{2^s}\right)$ (resp. $B\left(\widetilde{\mathcal{H}}\right)$).

3.33. There is a subalgebra $\widetilde{A}_{\text{smooth}} \subset \widetilde{A}_\pi$ generated by smooth elements. For any $s > 0$ there is a seminorm $\|\cdot\|_s$ on $\widetilde{A}_{\text{smooth}}$ given by

$$\|\widetilde{a}\|_s = \|\widetilde{a}^s\| = \left\|\lim_{n\to\infty} 1_{M(\widetilde{A})} \otimes \pi_n^s\left(a_n\right)\right\|. \tag{3.11}$$

Definition 3.34. The completion of $\widetilde{A}_{\text{smooth}}$ in the topology induced by seminorms $\|\cdot\|_s$ is said to be a *smooth algebra* of the coherent sequence (3.10) of spectral triples *(with respect to π)*. This algebra is denoted by $\widetilde{\mathcal{A}}_\pi$. We say that the sequence of spectral triple is *good* if $\widetilde{\mathcal{A}}_\pi$ is dense in $\widetilde{A}_\pi$. If π is an universal representation than "with respect to π is dropped and we write $\widetilde{\mathcal{A}}$ instead of $\widetilde{\mathcal{A}}_\pi$.

3.35. For any $\widetilde{a} \in \widetilde{W}^\infty_\pi$ we denote by

$$\widetilde{a}_D = \lim_{n\to\infty} 1_{M(\widetilde{A})} \otimes \left[D_n, \sum_{g\in J_n} \widetilde{a}\right] = \sum_{j=1}^k \widetilde{a}^j_D \otimes \omega_j \in L^2\left(\widetilde{A}_\pi\right) \otimes_A \Omega^1_D. \tag{3.12}$$

If $\mathcal{H}^\infty = \bigcap_{n=0}^\infty \operatorname{Dom} D^n$ then for any $\widetilde{a} \otimes \xi \in \widetilde{W}^\infty_\pi \otimes_{\mathcal{A}} \mathcal{H}^\infty$ we denote by

$$\widetilde{D}\left(\widetilde{a} \otimes \xi\right) \overset{\text{def}}{=} \sum_{j=1}^k \widetilde{a}^j_D \otimes \omega_j\left(\xi\right) + \widetilde{a} \otimes D\xi \in L^2\left(\widetilde{A}_\pi\right) \otimes_A \mathcal{H}, \tag{3.13}$$

i.e. $\widetilde{D}$ is a $\mathbb{C}$-linear map from $W^\infty_\pi \otimes_{\mathcal{A}} \mathcal{H}^\infty$ to $L^2\left(\widetilde{A}_\pi\right) \otimes_A \mathcal{H}$. The space $\widetilde{W}^\infty_\pi \otimes_{\mathcal{A}} \mathcal{H}^\infty$ is dense in $\widetilde{\mathcal{H}}$, hence the operator $\widetilde{D}$ can be regarded as an unbounded operator on $\widetilde{\mathcal{H}}$.

Definition 3.36. The coherent sequence (3.10) of spectral triples is said to be *regular (with respect to π)* if $\widetilde{\mathcal{A}}_\pi$ is a dense subalgebra of $\widetilde{A}_\pi$ in the C^*-norm topology. If π is the universal representation then "with respect to π" is dropped.

Definition 3.37. Let (3.10) be a regular coherent sequence of spectral triples. Let $\mathfrak{S} \in \mathfrak{FinAlg}$ is given by (3.9), and let $\widetilde{D}$ be given by (3.13). Let $\pi : \varinjlim A_n \to B\left(\mathcal{H}_\pi\right)$ be a good representation. Let $\left(A, \widetilde{A}_\pi, G_\pi\right)$ be the infinite noncommutative covering (with respect to π) of $\mathfrak{S}$. We say that $\left(\widetilde{\mathcal{A}}, \widetilde{\mathcal{H}}, \widetilde{D}\right)$ is a $\left(A, \widetilde{A}_\pi, G_\pi\right)$*-lift* of $\left(\mathcal{A}, \mathcal{H}, D\right)$.

4 Coverings of commutative spectral triples

The Spin-manifold is a Riemannian manifold M with a linear Spin-bundle S described in [11,21]. The bundle S is Hermitian. Taking into account than any Riemannian manifold has a natural measure μ, there is a Hilbert space $L^2\left(M, S\right) = L^2\left(M, S, \mu\right)$ described in 1.14. If $\Gamma^\infty\left(M, S\right)$ is a $C^\infty\left(M\right)$-module of smooth sections then there is a first order differential operator $\not{D}$ on $\Gamma^\infty\left(M, S\right)$. Locally $\not{D}$ is given by

$$\not{D}\xi = \sum_{j=1}^n \gamma_j\left(x\right) \frac{\partial}{\partial x_j}$$

where x_j $(j = 1, ..., n)$ are local coordinates on M, $\gamma_j \in \operatorname{End}_{C^\infty(M)}\left(\Gamma^\infty\left(M, S\right)\right)$ are described in [11,21]. Since $\Gamma^\infty\left(M, S\right)$ is a dense $\mathbb{C}$-subspace of $L^2\left(M, S\right)$ operator $\not{D}$ can be regarded as an unbounded operator $L^2\left(M, S\right)$. It is shown in [11,21] that $\left(C^\infty(M), L^2(M, S), \not{D}\right)$ is a spectral triple. For any $s \in \mathbb{N}^0$ there is a representation of $\pi^s : C^\infty\left(\widetilde{M}\right) \to B\left(L^2\left(M, S\right)^{2^s}\right)$ given by (1.4).

4.1 Finite-fold coverings

This section contains an algebraic version of the Proposition 1.1 in case of finite-fold coverings.

4.1.1 Coverings of C^*-algebras

Following two theorem state equivalence between a topological notion of a covering and an algebraical one.

Theorem 4.1. *[18] Suppose $\mathcal{X}$ and $\mathcal{Y}$ are compact Hausdorff connected spaces and $p : \mathcal{Y} \to \mathcal{X}$ is a continuous surjection. If $C(\mathcal{Y})$ is a projective finitely generated Hilbert module over $C(\mathcal{X})$ with respect to the action*

$$(f\xi)(y) = f(y)\xi(p(y)),\ f \in C(\mathcal{Y}),\ \xi \in C(\mathcal{X}),$$

then p is a finite-fold covering.

Theorem 4.2. *[12] If $\mathcal{X}$, $\widetilde{\mathcal{X}}$ are locally compact spaces, and $\pi : \widetilde{\mathcal{X}} \to \mathcal{X}$ is a surjective continuous map, then following conditions are equivalent:*

(i) The map $\pi : \widetilde{\mathcal{X}} \to \mathcal{X}$ is a finite-fold covering with a compactification,

(ii) There is a natural noncommutative finite-fold covering $\left(C_0\left(\mathcal{X}\right), C_0\left(\widetilde{\mathcal{X}}\right), G\right)$.

Remark 4.3. The definition of coverings with compactifications is presented in [12].

4.1.2 Topological coverings of spectral triples

Let $\left(C^\infty\left(M\right), L^2\left(M, S\right), \not{D}\right)$ be a commutative spectral triple, and let $\pi : \widetilde{M} \to M$ be a finite fold covering projection. From the Proposition 1.1 it follows that $\widetilde{M}$ has natural structure of the Riemannian manifold. Denote by $\widetilde{S} = \pi^* S$ the inverse image of the Spin-bundle S (cf. 1.2.2). Similarly we can define the inverse image $\widetilde{\not{D}} = \pi^* \not{D}$ (cf. Definition 1.13), and $\left(C^\infty\left(\widetilde{M}\right), L^2\left(\widetilde{M}, \widetilde{S}\right), \widetilde{\not{D}}\right)$ is a spectral triple. We would like to proof that

$$\left(C^\infty\left(\widetilde{M}\right), L^2\left(\widetilde{M}, \widetilde{S}\right), \widetilde{\not{D}}\right)$$

is the $\left(C\left(M\right), C\left(\widetilde{M}\right), G\left(\widetilde{M} \mid M\right)\right)$-lift of $\left(C^\infty\left(M\right), L^2\left(M, S\right), \not{D}\right)$.

4.1.3 Induced representation

Let us consider a family of open subsets $\left\{\mathcal{U}_\iota \subset M\right\}_{\iota \in I}$ such that

- $\mathcal{U}_\iota$ is evenly covered by π,
- The bundle S is trivial on $\mathcal{U}_\iota$.

The space M is compact, so there is a finite subfamily $\{\mathcal{U}_\iota \subset M\}_{\iota \in I}$ such that $M = \bigcup_{\iota \in I} \mathcal{U}_\iota$.

Proposition 4.4. *[4] A differential manifold M admits a (smooth) partition of unity if and only if it is paracompact.*

From the Proposition 4.4 it follows that there is a finite family $\{e_\iota \in C^\infty(M)\}_{\iota \in I}$ such that

$$e_\iota \left(M \backslash \mathcal{U}_\iota\right) = \{0\},$$
$$1_{C(M)} = \sum_{\iota \in I} e_\iota.$$

For any $\iota \in I$ there are smooth sections $\xi_\iota^1, ..., \xi_\iota^{\dim S} \in \Gamma^\infty(M, S)$ such that for any $x \in \mathcal{U}_\iota$ the set $\left\{\xi_{\iota,x}^1, ..., \xi_{\iota,x}^{\dim S}\right\} \subset \Gamma_x$ is a basis of Γ_x. If $s \in S$ is a section then

$$\xi = \sum_{\iota \in I} \xi_\iota, \text{ where } \xi_\iota = e_\iota \xi.$$

Otherwise there is unambiguous representation

$$\xi_\iota = \sum_{j=1}^{\dim S} e_\iota a_\iota^j \xi_\iota^j$$

where $a_\iota^j \in C^\infty(M)$. In fact any $\xi \in \Gamma^\infty(M, S)$ can be uniquely represented by the following sum

$$\xi = \sum_{\iota \in I} e_\iota \sum_{j=1}^{\dim S} a_\iota^j \xi_\iota^j. \tag{4.1}$$

For any $\iota \in I$ we can select an open connected subset $\widetilde{\mathcal{U}}_\iota \subset \widetilde{M}$ such that $\widetilde{\mathcal{U}}_\iota$ is homeomorphically mapped onto $\mathcal{U}_\iota$. If $\widetilde{e}_\iota \in C^\infty\left(\widetilde{M}\right)$ is given by

$$\widetilde{e}_\iota(\widetilde{x}) = \begin{cases} e_\iota(\pi(\widetilde{x})) & \widetilde{x} \in \widetilde{\mathcal{U}}_\iota \\ 0 & \widetilde{x} \notin \widetilde{\mathcal{U}}_\iota \end{cases}.$$

then

$$1_{C(\widetilde{M})} = \sum_{g \in G} \sum_{\iota \in I} g \widetilde{e}_\iota.$$

Moreover there for any $\iota \in I$ there are sections $\widetilde{\xi}_\iota^1, ..., \widetilde{\xi}_\iota^{\dim S} \in \Gamma^\infty\left(\widetilde{M}, \widetilde{S}\right)$ which are lifts of $\xi_\iota^1, ..., \xi_\iota^{\dim S}$. Similarly to the above construction any element in $\Gamma^\infty(M, S)$ can be represented as

$$\xi = \sum_{g \in G} \sum_{\iota \in I} g \widetilde{e}_\iota \sum_{j=1}^{\dim S} a_{(g,\iota)}^j \left(g \widetilde{\xi}_\iota^j\right) \text{ where } a_{(g,\iota)}^j \in C^\infty(M). \tag{4.2}$$

Now we can establish isomorphism

$$\Gamma^\infty\left(\widetilde{M},\widetilde{S}\right)\xrightarrow{\approx}C^\infty\left(\widetilde{M}\right)\otimes_{C^\infty(M)}\Gamma^\infty\left(M,S\right),$$
$$\sum_{g\in G}\sum_{\iota\in I}g\widetilde{e}_\iota\sum_{j=1}^{\dim S}a^j_{(g,\iota)}\left(g\widetilde{\xi}^j_\iota\right)\mapsto\sum_{g\in G}\sum_{\iota\in I}g\widetilde{e}_\iota\otimes\sum_{j=1}^{\dim S}a^j_{(g,\iota)}\xi^j_\iota.$$

Since $C^\infty\left(\widetilde{M}\right)$ is dense in $C\left(\widetilde{M}\right)$, and $\Gamma^\infty\left(M,S\right)$, (resp. $\Gamma^\infty\left(\widetilde{M},\widetilde{S}\right)$) is dense in $L^2\left(M,S\right)$, (resp. $L^2\left(\widetilde{M},\widetilde{S}\right)$) above isomorphism can be uniquely extended up to $\mathbb{C}$-isomorphism

$$L^2\left(\widetilde{M},\widetilde{S}\right)\xrightarrow{\approx}C\left(\widetilde{M}\right)\otimes_{C(M)}L^2\left(M,S\right).$$

Above formula coincides with construction 2.9 of induced representation. If $\widetilde{a}\otimes\xi,\widetilde{b}\otimes\eta\in C\left(\widetilde{M}\right)\otimes_{C(M)}\Gamma\left(M,S\right)\subset L^2\left(\widetilde{M},\widetilde{S}\right)$, μ (resp. $\widetilde{\mu}$) is the Riemannian measure (cf. [9]) on M, (resp. $\widetilde{M}$) then

$$\left(\widetilde{a}\otimes\xi,\widetilde{b}\otimes\eta\right)_{L^2\left(\widetilde{M},\widetilde{S}\right)}=\int_{\widetilde{M}}\widetilde{a}^*\left(\widetilde{x}\right)\widetilde{b}\left(\widetilde{x}\right)\left(\xi,\eta\right)_{\pi(\widetilde{x})}d\widetilde{\mu}=$$
$$=\int_M\sum_{g\in G\left(\widetilde{M}\mid M\right)}\left(g\left(\widetilde{a}^*\widetilde{b}\right)\right)\left(x\right)\left(\xi,\eta\right)_x d\mu=\left(\xi,\left\langle\widetilde{a},\widetilde{b}\right\rangle_{C\left(\widetilde{M}\right)}\eta\right)_{L^2(M,S)}.$$

Above equation is a version of (2.6). So the representation $\widetilde{\rho}:C\left(\widetilde{M}\right)\to L^2\left(\widetilde{M},\widetilde{S}\right)$ is induced by the pair $\left(C\left(M\right)\to L^2\left(M,S\right),\left(C\left(M\right),C\left(\widetilde{M}\right),G\left(\widetilde{M}\mid M\right)\right)\right)$.

4.1.4 Coverings of spectral triples

Operator $\widetilde{D\!\!\!/}$ (which is an inverse image of $D\!\!\!/$) can be regarded as unbounded operator on $L^2\left(\widetilde{M},\widetilde{S}\right)$, and satisfies to conditions (b) and (c) of the Definition 2.14. Clearly $C\left(\widetilde{M}\right)$ is a C^*-norm completion of $C^\infty\left(\widetilde{M}\right)$, i.e. condition (a) of the Definition 2.14 holds. In result we have the following theorem.

Theorem 4.5. *A spectral triple* $\left(C^\infty\left(\widetilde{M}\right),L^2\left(\widetilde{M},\widetilde{S}\right),\widetilde{D\!\!\!/}\right)$ *is a* $\left(C\left(M\right),C\left(\widetilde{M}\right),G\left(\widetilde{M}\mid M\right)\right)$-*lift of* $\left(C^\infty\left(M\right),L^2\left(M,S\right),D\!\!\!/\right)$.

4.2 Infinite coverings

This section contains an algebraic version of the Proposition 1.1 in case of infinite coverings.

4.2.1 Coverings of C^*-algebras

Following theorem states an equivalence between a topological notion of an infinite covering and an algebraical one.

Theorem 4.6. *[12] If* $\mathfrak{S}_{\mathcal{X}} = \{\mathcal{X} = \mathcal{X}_0 \leftarrow ... \leftarrow \mathcal{X}_n \leftarrow ...\} \in \mathfrak{FinTop}$ *and*

$$\mathfrak{S}_{C_0(\mathcal{X})} = \{C_0(\mathcal{X}) = C_0(\mathcal{X}_0) \to ... \to C_0(\mathcal{X}_n) \to ...\} \in \mathfrak{FinAlg}$$

is an algebraical finite covering sequence then following conditions hold:

(i) $\mathfrak{S}_{C_0(\mathcal{X})}$ *is good,*

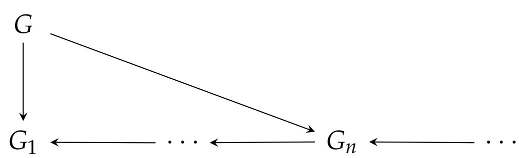
(ii) There are isomorphisms:

- $\varprojlim \downarrow \mathfrak{S}_{C_0(\mathcal{X})} \approx C_0\left(\varprojlim \downarrow \mathfrak{S}_{\mathcal{X}}\right)$;
- $G\left(\varprojlim \downarrow \mathfrak{S}_{C_0(\mathcal{X})} \mid C_0\left(\mathcal{X}\right)\right) \approx G\left(\varprojlim \downarrow \mathfrak{S}_{\mathcal{X}} \mid \mathcal{X}\right)$.

(cf. Definitions 1.35 and 3.14 for notation).

4.2.2 The sequence of spectral triples

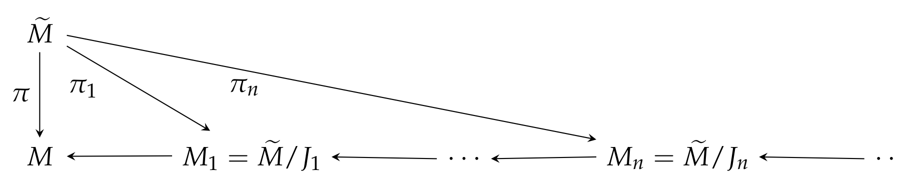
Let $\left(C^\infty\left(M\right), L^2\left(M, S\right), \not{D}\right)$ be a commutative spectral triple, and let $\pi : \widetilde{M} \to M$ be an infinite regular covering. From the Proposition 1.1 it follows that $\widetilde{M}$ has natural structure of the Riemannian manifold. Denote by $\widetilde{S} = \pi^* S$ the inverse image of the Spin-bundle (cf. 1.2.2). Similarly we can define an inverse image $\widetilde{\not{D}} = \pi^* \not{D}$ (cf. Definition 1.13). Let $G = G\left(\widetilde{M} \mid M\right)$ be a group of covering transformations of π. Suppose that there is a commutative diagram of group epimorphisms



such that

- A group G_n is finite for any $n \in \mathbb{N}$,
- $\bigcap_{n \in \mathbb{N}} \ker\left(G \to G_n\right)$ is a trivial group.

If $J_n = \ker\left(G \to G_n\right)$ then there is the following commutative diagram of coverings



Clearly $\mathfrak{S}_M = \{M = M_0 \leftarrow ... \leftarrow M_n \leftarrow ...\} \in \mathfrak{FinTop}$ is an topological finite covering sequence. From the Theorem 4.6 it turns out that

$$\mathfrak{S}_{C(M)} = \{C(M) = C(M_0) \to ... \to C(M_n) \to ...\}$$

is a good algebraical finite covering sequence, and the triple

$$\left(C(M), C_0\left(\widetilde{M}\right), G = G\left(\widetilde{M} \mid M\right)\right)$$

is an infinite noncommutative covering of $\mathfrak{S}_{C(M)}$. Otherwise from the Theorem 4.5 it follows that

$$\begin{split}\mathfrak{S}_{(C^\infty(M), L^2(M,S), \not{D})} = \{\left(C^\infty(M), L^2(M,S), \not{D}\right) = \left(C^\infty(M_0), L^2(M_0, S_0), \not{D}_0\right), \dots, \\ \left(C^\infty(M_n), L^2(M_n, S_n), \not{D}_n\right), \dots\} \in \mathfrak{CohTriple}\end{split} \tag{4.3}$$

is a coherent sequence of spectral triples. We would like to proof that $\mathfrak{S}_{(C^\infty(M), L^2(M,S), \not{D})}$ is regular and to find a $\left(C(M), C_0\left(\widetilde{M}\right), G\right)$-lift of $\left(C^\infty(M), L^2(M,S), \not{D}\right)$. Denote by $\widetilde{S} = \pi^* S$ the inverse image of the Spin-bundle S.

4.2.3 Induced representation

Similarly to 4.1.3 consider a finite family a finite family $\{e_\iota \in C^\infty(M)\}_{\iota \in I}$ of positive elements such that

$$e_\iota(M \backslash \mathcal{U}_\iota) = \{0\},$$
$$1_{C(M)} = \sum_{\iota \in I} e_\iota.$$

and smooth sections $\xi_\iota^1, ..., \xi_\iota^{\dim S} \in \Gamma^\infty(M, S)$ such that for any $x \in \mathcal{U}_\iota$ the set $\left\{\xi_{\iota,x}^1, ..., \xi_{\iota,x}^{\dim S}\right\} \subset \Gamma_x$ is a basis of Γ_x. Similarly to 4.1.3 we define $\widetilde{e}_\iota \in C^\infty\left(\widetilde{M}\right)$ is given by

$$\widetilde{e}_\iota(\widetilde{x}) = \begin{cases} e_\iota(\pi(\widetilde{x})) & \widetilde{x} \in \widetilde{\mathcal{U}}_\iota \\ 0 & \widetilde{x} \notin \widetilde{\mathcal{U}}_\iota \end{cases}.$$

Clearly that elements $\widetilde{e}_\iota$ are smooth. Similarly to (4.2) we have

$$\widetilde{\xi} = \sum_{g \in G} \sum_{\iota \in I} g \widetilde{e}_\iota \sum_{j=1}^{\dim S} a_{(g,\iota)}^j \left(g \widetilde{\xi}_\iota^j\right) \text{ where } a_{(g,\iota)}^j \in C^\infty(M). \tag{4.4}$$

However there is substantial difference between (4.2) and (4.4) because first equation operates with finite group G and second one with infinite one. So (4.4) can be regarded as a point-wise limit. If we regard different finite subsets $H \subset G$ we obtain finite sums

$$\xi = \sum_{g \in H} \sum_{\iota \in I} g \widetilde{e}_\iota \sum_{j=1}^{\dim S} a_{(g,\iota)}^j \left(g \widetilde{\xi}_\iota^j\right).$$

The space of above finite sums coincides with the subspace of sections $\Gamma^{\infty}_c\left(\widetilde{M},\widetilde{S}\right) \subset \Gamma^{\infty}\left(\widetilde{M},\widetilde{S}\right)$ with compact support. Denote by $L^2\left(C_0\left(\widetilde{M}\right)\right) \subset C_0\left(\widetilde{M}\right)$ the space of square-summable elements (cf. Definition 3.19). From $C_c\left(\widetilde{M}\right) \subset L^2\left(C_0\left(\widetilde{M}\right)\right)$ and since $C_c\left(\widetilde{M}\right)$ is dense in $C_0\left(\widetilde{M}\right)$ it turns out $C_c\left(\widetilde{M}\right)$ is dense in $L^2\left(C_0\left(\widetilde{M}\right)\right)$. There is a $\mathbb{C}$-linear isomorphism

$$\Gamma_c\left(\widetilde{M},\widetilde{S}\right) \to C_c\left(\widetilde{M}\right) \otimes_{C(M)} \Gamma\left(M,S\right),$$
$$\sum_{g\in G}\sum_{\iota\in I} g\widetilde{e}_\iota \sum_{j=1}^{\dim S} a^j_{(g,\iota)}\left(g\widetilde{\xi}^j_\iota\right) \mapsto \sum_{g\in G}\sum_{\iota\in I} g\widetilde{e}_\iota \otimes \sum_{j=1}^{\dim S} a^j_{(g,\iota)}\xi^j_\iota$$
$$\text{where the set } \left\{g\in G \mid a^j_{(g,\iota)} \neq 0\right\} \text{ is finite.}$$

If $\widetilde{a}\otimes\xi, \widetilde{b}\otimes\eta \in C_c\left(\widetilde{M}\right) \otimes_{C(M)} \Gamma\left(M,S\right) \subset L^2\left(\widetilde{M},\widetilde{S}\right)$, μ (resp. $\widetilde{\mu}$) is the Riemannian measure on M, (resp. $\widetilde{M}$) then

$$\left(\widetilde{a}\otimes\xi,\widetilde{b}\otimes\eta\right)_{L^2\left(\widetilde{M},\widetilde{S}\right)} = \int_{\widetilde{M}} \widetilde{a}^*\left(\widetilde{x}\right)\widetilde{b}\left(\widetilde{x}\right)\left(\xi,\eta\right)_{\pi(\widetilde{x})} d\widetilde{\mu} =$$
$$= \int_M \sum_{g\in G\left(\widetilde{M}\mid M\right)} \left(g\left(\widetilde{a}^*\widetilde{b}\right)\right)(x)\left(\xi,\eta\right)_x d\mu = \left(\xi, \left\langle\widetilde{a},\widetilde{b}\right\rangle_{C_0\left(\widetilde{M}\right)}\eta\right)_{L^2(M,S)}.$$

Above equation is a version of the scalar product given by (3.7). The space $\Gamma_c\left(\widetilde{M},\widetilde{S}\right)$ is dense in $L^2\left(\widetilde{M},\widetilde{S}\right)$ and $C_c\left(\widetilde{M}\right)$ is dense in $L^2\left(C_0\left(\widetilde{M}\right)\right)$. It follows that $L^2\left(\widetilde{M},\widetilde{S}\right)$ is the Hilbert completion of $C_c\left(\widetilde{M}\right) \otimes_{C(M)} L^2\left(M,S\right)$ or, equivalently, $L^2\left(\widetilde{M},\widetilde{S}\right)$ is the Hilbert completion of $L^2\left(C_0\left(\widetilde{M}\right)\right) \otimes_{C(M)} L^2\left(M,S\right)$. So if $\rho: C\left(M\right) \to L^2\left(M,S\right)$ then the representation $\widetilde{\rho}: C_0\left(\widetilde{M}\right) \to L^2\left(\widetilde{M},\widetilde{S}\right)$ is induced by the pair

$$\left(\rho, \left(C\left(M\right), C_0\left(\widetilde{M}\right), G\left(\widetilde{M}\mid M\right)\right)\right).$$

4.2.4 Coverings of spectral triples

Consider the coherent sequence

$$\begin{aligned}\mathfrak{S}_{\left(C^\infty(M),L^2(M,S),\not{D}\right)} = \{&\left(C^\infty\left(M\right), L^2\left(M,S\right), \not{D}\right) = \left(C^\infty\left(M_0\right), L^2\left(M_0,S_0\right), \not{D}_0\right), \dots, \\ &\left(C^\infty\left(M_n\right), L^2\left(M_n,S_n\right), \not{D}_n\right), \dots\} \in \mathfrak{CohTriple}\end{aligned} \tag{4.5}$$

of spectral triples given by (4.3). If $\widetilde{W}^\infty$ is a space $\mathfrak{S}_{\left(C^\infty(M),L^2(M,S),\not{D}\right)}$-smooth elements then from the condition (d) of the Definition 3.31 it follows that $\widetilde{W}^\infty \subset K\left(C_0\left(\widetilde{M}\right)\right) = C_c\left(\widetilde{M}\right)$. For any $n\in\mathbb{N}$ denote by $\widetilde{\pi}_n: \widetilde{M} \to M_n$ the natural covering.

Lemma 4.7. *[12] Following conditions hold:*

(i) If $\widetilde{\mathcal{U}} \subset \widetilde{M}$ is a compact set then there is $N \in \mathbb{N}$ such that for any $n \geq N$ the restriction $\widetilde{\pi}_n|_{\widetilde{\mathcal{U}}} : \widetilde{\mathcal{U}} \xrightarrow{\approx} \widetilde{\pi}_n\left(\widetilde{\mathcal{U}}\right)$ is a homeomorphism,

(ii) If $\widetilde{a} \in C_c\left(\widetilde{M}\right)_+$ is a positive element then there there is $N \in \mathbb{N}$ such that for any $n \geq \mathbb{N}$ following condition holds

$$a_n\left(\widetilde{\pi}_n\left(\widetilde{x}\right)\right) = \begin{cases} \widetilde{a}\left(\widetilde{x}\right) & \widetilde{x} \in \mathfrak{supp}\ \widetilde{a}\ \&\ \widetilde{\pi}_n\left(\widetilde{x}\right) \in \mathfrak{supp}\ a_n \\ 0 & \widetilde{\pi}_n\left(\widetilde{x}\right) \notin \mathfrak{supp}\ a_n \end{cases} \tag{4.6}$$

where

$$a_n = \sum_{g \in \ker\left(\widehat{G} \to G_n\right)} g\widetilde{a}.$$

Lemma 4.8. *Following condition holds*

$$\widetilde{W}^\infty \subset C_c^\infty\left(\widetilde{M}\right) \stackrel{\text{def}}{=} C^\infty\left(\widetilde{M}\right) \bigcap C_c\left(\widetilde{M}\right).$$

Proof. If $\widetilde{a} \in \widetilde{W}^\infty$ then from the condition (a) of the Definition 3.31 it follows that

$$a_n = \sum_{g \in \ker\left(\widehat{G} \to G_n\right)} g\widetilde{a} \in C^\infty\left(M_n\right).$$

From $\widetilde{W}^\infty \subset C_c\left(\widetilde{M}\right)$ and (4.6) it follows that $\widetilde{a} \in C^\infty\left(\widetilde{M}\right)$, hence $\widetilde{a} \in C^\infty\left(\widetilde{M}\right) \bigcap C_c\left(\widetilde{M}\right)$. □

Lemma 4.9. *Following condition holds*

$$C_c^\infty\left(\widetilde{M}\right) \subset \widetilde{W}^\infty.$$

Proof. Let $\widetilde{a} \in C_c^\infty\left(\widetilde{M}\right)$, Let $\pi_n^s : C^\infty\left(M_n\right) \to B\left(\mathcal{H}_n^{2^s}\right)$ be a representation given by (1.4). From (4.6) it turns out

$$\lim_{n \to \infty} 1_{C_b\left(\widetilde{M}\right)} \otimes \pi_n^s\left(a_n\right) = \pi_s\left(\widetilde{a}\right) \tag{4.7}$$

in sense of strong convergence. The equation (4.7) means that any $\widetilde{a} \in C_c^\infty\left(\widetilde{M}\right)$ satisfies to the condition (b) the Definition (3.31). The manifold is compact, so there is a finite set $\{\mathcal{U}_\iota\}_{\iota \in I}$ of open sets such that any $\mathcal{U}_\iota$ is evenly covered by $\widetilde{M} \to M$ and $M = \bigcup_{\iota \in I} \mathcal{U}_\iota$. There is a partition of unity

$$1_{C(M)} = \sum_{\iota \in I} a_\iota; \text{ where } a_\iota \in C^\infty\left(M\right). \tag{4.8}$$

For any $\iota \in I$ we select $\widetilde{\mathcal{U}}_\iota$ such that the natural map $\widetilde{\mathcal{U}}_\iota \to \mathcal{U}_\iota$ is a homeomorphism. Denote by $e_\iota = \sqrt{a_\iota}$ and let $\widetilde{e}_\iota \in C_c^\infty\left(\widetilde{M}\right)$ is given by

$$\widetilde{e}_\iota\left(\widetilde{x}\right) = \begin{cases} e_\iota\left(\widetilde{\pi}\left(\widetilde{x}\right)\right) & \widetilde{x} \in \widetilde{\mathcal{U}}_\iota \\ 0 & \widetilde{x} \notin \widetilde{\mathcal{U}}_\iota \end{cases}$$

From (4.8) it follows that

$$\widetilde{a} = \sum_{(g,\iota)\in G\left(\widetilde{M}\ \mid\ M\right)\times I} \left(ge_\iota\right)\left(ge_\iota\right)\widetilde{a} \tag{4.9}$$

or, equivalently, if $\widetilde{\alpha}_{(g,e_\iota)} = ge_\iota$ and $\widetilde{\beta}_{(g,e_\iota)} = \left(ge_\iota\right)\widetilde{a}$ then

$$\widetilde{a} = \sum_{(g,\iota)\in G\left(\widetilde{M}\ \mid\ M\right)\times I} \widetilde{a}_{(g,e_\iota)};\ \text{where } \widetilde{a}_{(g,e_\iota)} = \widetilde{\alpha}_{(g,e_\iota)}\widetilde{\beta}_{(g,e_\iota)}. \tag{4.10}$$

The set $\mathfrak{supp}\ \widetilde{a}$ is compact, it follows that there is a finite subset $\widetilde{I} \subset G\left(\widetilde{M}\ \mid\ M\right) \times I$ such that

$$\mathfrak{supp}\ \widetilde{a} \bigcap \left(\bigcup_{(g,e_\iota)\in\left(G\left(\widetilde{M}\ \mid\ M\right)\times I\right)\setminus\widetilde{I}} g\widetilde{\mathcal{U}}_\iota\right) = \emptyset,$$

and taking into account (4.10) one has

$$\widetilde{a} = \sum_{(g,\iota)\in\widetilde{I}} \widetilde{a}_{(g,e_\iota)} = \sum_{(g,\iota)\in\widetilde{I}} \widetilde{\alpha}_{(g,e_\iota)}\widetilde{\beta}_{(g,e_\iota)}. \tag{4.11}$$

It follows that from (4.11).

$$\begin{aligned}\left[\widetilde{\not{D}}, \widetilde{a}\right] &= \sum_{(g,\iota)\in\widetilde{I}} \left[\widetilde{\not{D}}, \widetilde{\alpha}_{(g,e_\iota)}\right]\widetilde{\beta}_{(g,e_\iota)} + \widetilde{\alpha}_{(g,e_\iota)}\left[\widetilde{\not{D}}, \widetilde{\beta}_{(g,e_\iota)}\right] = \\ &= \sum_{(g,\iota)\in\widetilde{I}} \widetilde{\beta}_{(g,e_\iota)}\left[\widetilde{\not{D}}, \widetilde{\alpha}_{(g,e_\iota)}\right] + \widetilde{\alpha}_{(g,e_\iota)}\left[\widetilde{\not{D}}, \widetilde{\beta}_{(g,e_\iota)}\right]\end{aligned} \tag{4.12}$$

where the equality $\left[\widetilde{\not{D}}, \widetilde{\alpha}_{(g,e_\iota)}\right]\widetilde{\beta}_{(g,e_\iota)} = \widetilde{\beta}_{(g,e_\iota)}\left[\widetilde{\not{D}}, \widetilde{\alpha}_{(g,e_\iota)}\right]$ follows from commutativity of $C^\infty\left(\widetilde{M}\right)$. If $a_{(g,e_\iota)}, \alpha_{(g,e_\iota)}, \beta_{(g,e_\iota)} \in C^\infty(M)$ are given by

$$a_{(g,e_\iota)}\left(\widetilde{\pi}\left(\widetilde{x}\right)\right) = \begin{cases} \widetilde{a}_{(g,e_\iota)}\left(\widetilde{x}\right) & \widetilde{x} \in g\widetilde{\mathcal{U}}_\iota \\ 0 & \widetilde{\pi}\left(\widetilde{x}\right) \notin \mathcal{U}_\iota \end{cases},$$

$$\alpha_{(g,e_\iota)}\left(\widetilde{\pi}\left(\widetilde{x}\right)\right) = \begin{cases} \widetilde{\alpha}_{(g,e_\iota)}\left(\widetilde{x}\right) & \widetilde{x} \in g\widetilde{\mathcal{U}}_\iota \\ 0 & \widetilde{\pi}\left(\widetilde{x}\right) \notin \mathcal{U}_\iota \end{cases},$$

$$\beta_{(g,e_\iota)}\left(\widetilde{\pi}\left(\widetilde{x}\right)\right) = \begin{cases} \widetilde{\beta}_{(g,e_\iota)}\left(\widetilde{x}\right) & \widetilde{x} \in g\widetilde{\mathcal{U}}_\iota \\ 0 & \widetilde{\pi}\left(\widetilde{x}\right) \notin \mathcal{U}_\iota \end{cases},$$

and $\chi_{(g,e_\iota)} \in C_0\left(\widetilde{M}\right)''$ is the characteristic function of $g\widetilde{\mathcal{U}}_\iota$ then from

$$\mathfrak{supp}\,\widetilde{a}_{(g,e_\iota)}, \mathfrak{supp}\,\widetilde{\alpha}_{(g,e_\iota)}, \mathfrak{supp}\,\widetilde{\beta}_{(g,e_\iota)} \subset g\widetilde{\mathcal{U}}_\iota$$

it follows that

$$\begin{split}\widetilde{\alpha}_{(g,e_\iota)} = \chi_{(g,e_\iota)}\alpha_{(g,e_\iota)} = \alpha_{(g,e_\iota)}\chi_{(g,e_\iota)} = \chi_{(g,e_\iota)}\alpha_{(g,e_\iota)}\chi_{(g,e_\iota)},\\ \left[\widetilde{\not{D}}, \widetilde{\alpha}_{(g,e_\iota)}\right] = \chi_{(g,e_\iota)}\left[\not{D}, \alpha_{(g,e_\iota)}\right] = \left[\not{D}, \alpha_{(g,e_\iota)}\right]\chi_{(g,e_\iota)} = \chi_{(g,e_\iota)}\left[\not{D}, \alpha_{(g,e_\iota)}\right]\chi_{(g,e_\iota)},\\ \widetilde{\beta}_{(g,e_\iota)} = \chi_{(g,e_\iota)}\beta_{(g,e_\iota)} = \beta_{(g,e_\iota)}\chi_{(g,e_\iota)} = \chi_{(g,e_\iota)}\beta_{(g,e_\iota)}\chi_{(g,e_\iota)},\\ \left[\widetilde{\not{D}}, \widetilde{\beta}_{(g,e_\iota)}\right] = \chi_{(g,e_\iota)}\left[\not{D}, \beta_{(g,e_\iota)}\right] = \left[\not{D}, \beta_{(g,e_\iota)}\right]\chi_{(g,e_\iota)} = \chi_{(g,e_\iota)}\left[\not{D}, \beta_{(g,e_\iota)}\right]\chi_{(g,e_\iota)}.\end{split} \tag{4.13}$$

From (4.12) and (4.13) it follows that

$$\begin{split}\left[\widetilde{\not{D}}, \widetilde{a}\right] = \sum_{(g,\iota)\in\widetilde{I}} \widetilde{\beta}_{(g,e_\iota)}\left[\not{D}, \chi_{(g,e_\iota)}\alpha_{(g,e_\iota)}\chi_{(g,e_\iota)}\right] + \widetilde{\alpha}_{(g,e_\iota)}\left[\not{D}, \chi_{(g,e_\iota)}\beta_{(g,e_\iota)}\chi_{(g,e_\iota)}\right] =\\ = \sum_{(g,\iota)\in\widetilde{I}} \widetilde{\beta}_{(g,e_\iota)}\chi_{(g,e_\iota)}\left[\not{D}, \alpha_{(g,e_\iota)}\right] + \widetilde{\alpha}_{(g,e_\iota)}\chi_{(g,e_\iota)}\left[\not{D}, \beta_{(g,e_\iota)}\right] =\\ = \sum_{(g,\iota)\in\widetilde{I}} \widetilde{\beta}_{(g,e_\iota)}\left[\not{D}, \alpha_{(g,e_\iota)}\right] + \widetilde{\alpha}_{(g,e_\iota)}\left[\not{D}, \beta_{(g,e_\iota)}\right].\end{split} \tag{4.14}$$

where $\not{D}$ is Dirac operator on M. If $\alpha^n_{(g,e_\iota)}, \beta^n_{(g,e_\iota)} \in C^\infty(M_n)$ are given by

$$\alpha^n_{(g,e_\iota)} = \sum_{g\in\ker\left(G\left(\widetilde{M}\mid M\right)\to G\left(\widetilde{M}\mid M_n\right)\right)} g\widetilde{\alpha}_{(g,e_\iota)},$$
$$\beta^n_{(g,e_\iota)} = \sum_{g\in\ker\left(G\left(\widetilde{M}\mid M\right)\to G\left(\widetilde{M}\mid M_n\right)\right)} g\widetilde{\beta}_{(g,e_\iota)},$$

then for any $n \geq N$ following condition holds

$$1_{C_b\left(\widetilde{M}\right)} \otimes [\not{D}_n, a_n] = \sum_{(g,\iota)\in\widetilde{I}} {\beta^n}_{(g,e_\iota)}\left[\not{D}, \alpha_{(g,e_\iota)}\right] + \alpha^n_{(g,e_\iota)}\left[\not{D}, \beta_{(g,e_\iota)}\right]$$

From the strong limits $\lim_{n\to\infty}\alpha_{(g,e_\iota)} = \widetilde{\alpha}_{(g,e_\iota)}$, $\lim_{n\to\infty}\beta_{(g,e_\iota)} = \widetilde{\beta}_{(g,e_\iota)}$ it follows that

$$\begin{split}\lim_{n\to\infty} 1_{C_b\left(\widetilde{M}\right)} \otimes [\not{D}_n, a_n] = \sum_{(g,\iota)\in\widetilde{I}} \left(\lim_{n\to\infty} {\beta^n}_{(g,e_\iota)}\left[\not{D}, \alpha_{(g,e_\iota)}\right] + \lim_{n\to\infty}\alpha^n_{(g,e_\iota)}\left[\not{D}, \beta_{(g,e_\iota)}\right]\right)\\ = \sum_{(g,\iota)\in\widetilde{I}} \left(\widetilde{\beta}_{(g,e_\iota)}\left[\not{D}, \alpha_{(g,e_\iota)}\right] + \widetilde{\alpha}_{(g,e_\iota)}\left[\not{D}, \beta_{(g,e_\iota)}\right]\right)\end{split} \tag{4.15}$$

where the above limit means the strong convergence. The right part of (4.15) lies in $C_c\left(\widetilde{M}\right) \otimes_{C_0(M)} \Omega^1_{\not{D}} \subset L^2\left(C_0\left(\widetilde{M}\right)\right) \otimes_{C_0(M)} \Omega^1_{\not{D}}$, it follows that $\widetilde{a}$ satisfies to the condition (c) of the Definition 3.31. It turns out any $\widetilde{a} \in C^\infty_c\left(\widetilde{M}\right)$ is $\mathfrak{S}_{(C^\infty(M), L^2(M,S), \not{D})}$-smooth. $\square$

If $C_0^\infty\left(\widetilde{M}\right)$ is the completion with respect to seminorms $\|\cdot\|_s$ given by (3.11) then $C_0^\infty\left(\widetilde{M}\right)$ is the smooth algebra of the coherent sequence (4.5) (cf. Definition 3.34). The algebra $C_0^\infty\left(\widetilde{M}\right)$ is a dense subalgebra of $C_0\left(\widetilde{M}\right)$ in the C^*-norm topology, so the sequence (4.5) is regular (cf. Definition 3.36). If $\widetilde{a} \in C_c^\infty\left(\widetilde{M}\right)$ and $\widetilde{a}_{\not D}$ is given by (3.12), i.e.

$$\widetilde{a}_{\not D} = \lim_{n\to\infty} 1_{M\left(\widetilde{A}\right)} \otimes \left[\not D_n, \sum_{g\in J_n} \widetilde{a}\right] \tag{4.16}$$

then from (4.15) it follows that

$$\widetilde{a}_{\not D} = \sum_{(g,\iota)\in\widetilde{I}} \widetilde{\beta}_{(g,e_\iota)} \otimes \left[\not D, \alpha_{(g,e_\iota)}\right] + \widetilde{\alpha}_{(g,e_\iota)} \otimes \left[\not D, \beta_{(g,e_\iota)}\right] \tag{4.17}$$

If $\xi \in \mathcal{H}^\infty = \Gamma^\infty(M, \mathcal{S})$ and $\not D'$ is given by (3.13), i.e.

$$\not D'\left(\widetilde{a}\otimes\xi\right) = \sum_{j=1}^{k} \widetilde{a}^j_{\not D} \otimes \omega_j(\xi) + \widetilde{a}\otimes\not D\xi =$$

$$= \sum_{(g,\iota)\in\widetilde{I}} \widetilde{\beta}_{(g,e_\iota)} \otimes \left[\not D, \alpha_{(g,e_\iota)}\right]\xi + \widetilde{\alpha}_{(g,e_\iota)} \otimes \left[\not D, \beta_{(g,e_\iota)}\right]\xi + \widetilde{a}\otimes\not D\xi.$$

then clearly

$$\not D'\left(\widetilde{a}\otimes\xi\right) = \widetilde{\not D}\left(\widetilde{a}\otimes\xi\right) \text{ where } \widetilde{\not D} \text{ is the lift of } \not D \text{ (cf. Definition 1.13).}$$

Since the space $C^\infty\left(\widetilde{M}\right)\bigcap C_c\left(\widetilde{M}\right)$ is dense in $C_0\left(\widetilde{M}\right)$, we have the following theorem

Theorem 4.10. *Following conditions hold:*

- *The sequence of spectral triples*

$$\mathfrak{S}_{(C^\infty(M),L^2(M,S),\not D)} = \left\{\left(C^\infty(M), L^2(M,S), \not D\right) = \left(C^\infty(M_0), L^2(M_0,S_0), \not D_0\right), \dots, \left(C^\infty(M_n), L^2(M_n,S_n), \not D_n\right), \dots\right\} \in \mathfrak{CohTriple}$$

  given by (4.3) *is regular (cf. Definition 3.36),*

- *The triple* $\left(C_0^\infty\left(\widetilde{M}\right), L^2\left(\widetilde{M}, \widetilde{\mathcal{S}}\right), \widetilde{\not D}\right)$ *is the* $\left(C(M), C_0\left(\widetilde{M}\right), G\left(\widetilde{M}\mid M\right)\right)$*-lift of*

$$\left(C^\infty(M), L^2(M,\mathcal{S}), \not D\right)$$

  (cf. Definition 3.37).

5 Coverings of noncommutative tori

5.1 Fourier transformation

There is a norm on $\mathbb{Z}^n$ given by

$$\|(k_1, ..., k_n)\| = \sqrt{k_1^2 + ... + k_n^2}.$$

The space of complex-valued Schwartz functions on $\mathbb{Z}^n$ is given by

$$\mathcal{S}(\mathbb{Z}^n) = \left\{a = \{a_k\}_{k\in\mathbb{Z}^n} \in \mathbb{C}^{\mathbb{Z}^n} \mid \sup_{k\in\mathbb{Z}^n} (1 + \|k\|)^s |a_k| < \infty, \ \forall s \in \mathbb{N}\right\}.$$

Let $\mathbb{T}^n$ be an ordinary n-torus. We will often use real coordinates for $\mathbb{T}^n$, that is, view $\mathbb{T}^n$ as $\mathbb{R}^n/\mathbb{Z}^n$. Let $C^\infty(\mathbb{T}^n)$ be an algebra of infinitely differentiable complex-valued functions on $\mathbb{T}^n$. There is the bijective Fourier transformations $\mathcal{F}_{\mathbb{T}} : C^\infty(\mathbb{T}^n) \xrightarrow{\approx} \mathcal{S}(\mathbb{Z}^n)$; $f \mapsto \widehat{f}$ given by

$$\widehat{f}(p) = \mathcal{F}_{\mathbb{T}}(f)(p) = \int_{\mathbb{T}^n} e^{-2\pi i x\cdot p} f(x)\, dx \tag{5.1}$$

where dx is induced by the Lebesgue measure on $\mathbb{R}^n$ and $\cdot$ is the scalar product on the Euclidean space $\mathbb{R}^n$. The Fourier transformation carries multiplication to convolution, i.e.

$$\widehat{fg}(p) = \sum_{r+s=p} \widehat{f}(r)\, \widehat{g}(s).$$

The inverse Fourier transformation $\mathcal{F}_{\mathbb{T}}^{-1} : \mathcal{S}(\mathbb{Z}^n) \xrightarrow{\approx} C^\infty(\mathbb{T}^n)$; $\widehat{f} \mapsto f$ is given by

$$f(x) = \mathcal{F}_{\mathbb{T}}^{-1}\widehat{f}(x) = \sum_{p\in\mathbb{Z}^n} \widehat{f}(p)\, e^{2\pi i x\cdot p}.$$

There is the $\mathbb{C}$-valued scalar product on $C^\infty(\mathbb{T}^n)$ given by

$$(f, g) = \int_{\mathbb{T}^n} fg dx = \sum_{p\in\mathbb{Z}^n} \widehat{f}(-p)\, \widehat{g}(p).$$

Denote by $\mathcal{S}(\mathbb{R}^n)$ be the space of complex Schwartz (smooth, rapidly decreasing) functions on $\mathbb{R}^n$.

$$\mathcal{S}(\mathbb{R}^n) = \left\{f \in C^\infty(\mathbb{R}^n) : \|f\|_{\alpha,\beta)} < \infty \quad \forall \alpha = (\alpha_1, ...\alpha_n), \beta = (\beta_1, ...\beta_n) \in \mathbb{Z}_+^n\right\},$$
$$\|f\|_{\alpha,\beta} = \sup_{x\in\mathbb{R}^n} \left|x^\alpha D^\beta f(x)\right| \tag{5.2}$$

where

$$x^\alpha = x_1^{\alpha_1} \cdot ... \cdot x_n^{\alpha_n},$$
$$D^\beta = \frac{\partial}{\partial x_1^{\beta_1}} ... \frac{\partial}{\partial x_n^{\beta_n}}.$$

The topology on $\mathcal{S}(\mathbb{R}^n)$ is given by seminorms $\|\cdot\|_{\alpha,\beta}$.

Definition 5.1. Denote by $\mathcal{S}'(\mathbb{R}^n)$ the vector space dual to $\mathcal{S}(\mathbb{R}^n)$, i.e. the space of continuous functionals on $\mathcal{S}(\mathbb{R}^n)$. Denote by $\langle\cdot,\cdot\rangle : \mathcal{S}'(\mathbb{R}^n) \times \mathcal{S}(\mathbb{R}^n) \to \mathbb{C}$ the natural pairing. We say that $\{a_n \in \mathcal{S}'(\mathbb{R}^n)\}_{n\in\mathbb{N}}$ is *weakly-* convergent* to $a \in \mathcal{S}'(\mathbb{R}^n)$ if for any $b \in \mathcal{S}(\mathbb{R}^n)$ following condition holds

$$\lim_{n\to\infty} \langle a_n, b\rangle = \langle a, b\rangle .$$

We say that

$$a = \lim_{n\to\infty} a_n$$

in the *sense of weak-* convergence*.

Let $\mathcal{F}$ and $\mathcal{F}^{-1}$ be the ordinary and inverse Fourier transformations given by

$$(\mathcal{F}f)(u) = \int_{\mathbb{R}^{2N}} f(t)e^{-2\pi it\cdot u}dt,\ \left(\mathcal{F}^{-1}f\right)(u) = \int_{\mathbb{R}^{2N}} f(t)e^{2\pi it\cdot u}dt \tag{5.3}$$

which satisfy following conditions

$$\mathcal{F}\circ\mathcal{F}^{-1}|_{\mathcal{S}(\mathbb{R}^n)} = \mathcal{F}^{-1}\circ\mathcal{F}|_{\mathcal{S}(\mathbb{R}^n)} = \mathrm{Id}_{\mathcal{S}(\mathbb{R}^n)}.$$

There is the $\mathbb{C}$-valued scalar product on $\mathcal{S}(\mathbb{R}^n)$ given by

$$(f,g) = \int_{\mathbb{R}^n} fg dx = \int_{\mathbb{R}^n} \mathcal{F}f\mathcal{F}g dx. \tag{5.4}$$

which if $\mathcal{F}$-invariant, i.e.

$$(f,g) = (\mathcal{F}f, \mathcal{F}g).$$

5.2 Noncommutative torus $\mathbb{T}^n_\Theta$

Let Θ be a real skew-symmetric $n\times n$ matrix, we will define a new noncommutative product $\star_\Theta$ on $\mathcal{S}(\mathbb{Z}^n)$ given by

$$\left(\widehat{f}\star_\Theta \widehat{g}\right)(p) = \sum_{r+s=p} \widehat{f}(r)\,\widehat{g}(s)\,e^{-\pi ir\,\cdot\,\Theta s}. \tag{5.5}$$

and an involution

$$\widehat{f}^*(p) = \overline{\widehat{f}}(-p)).$$

In result there is an involutive algebra $C^\infty\left(\mathbb{T}^n_\Theta\right) = \left(\mathcal{S}(\mathbb{Z}^n), +, \star_\Theta, ^*\right)$. There is a tracial state on $C^\infty\left(\mathbb{T}^n_\Theta\right)$ given by

$$\tau(f) = \widehat{f}(0). \tag{5.6}$$

From $C^\infty\left(\mathbb{T}^n_\Theta\right) \approx \mathcal{S}(\mathbb{Z}^n)$ it follows that there is a $\mathbb{C}$-linear isomorphism

$$\varphi_\infty : C^\infty\left(\mathbb{T}^n_\Theta\right) \xrightarrow{\approx} C^\infty\left(\mathbb{T}^n\right). \tag{5.7}$$

such that following condition holds

$$\tau(f) = \frac{1}{(2\pi)^n}\int_{\mathbb{T}^n} \varphi_\infty(f)\ dx. \tag{5.8}$$

Similarly to 1.46 there is the Hilbert space $L^2\left(C^\infty\left(\mathbb{T}^n_\Theta\right),\tau\right)$ and the natural representation $C^\infty\left(\mathbb{T}^n_\Theta\right)\to B\left(L^2\left(C^\infty\left(\mathbb{T}^n_\Theta\right),\tau\right)\right)$ which induces the C^*-norm. The C^*-norm completion $C\left(\mathbb{T}^n_\Theta\right)$ of $C^\infty\left(\mathbb{T}^n_\Theta\right)$ is a C^*-algebra and there is a faithful representation

$$C\left(\mathbb{T}^n_\Theta\right)\to B\left(L^2\left(C^\infty\left(\mathbb{T}^n_\Theta\right),\tau\right)\right). \tag{5.9}$$

We will write $L^2\left(C\left(\mathbb{T}^n_\Theta\right),\tau\right)$ instead of $L^2\left(C^\infty\left(\mathbb{T}^n_\Theta\right),\tau\right)$. There is the natural $\mathbb{C}$-linear map $C^\infty\left(\mathbb{T}^n_\Theta\right)\to L^2\left(C\left(\mathbb{T}^n_\Theta\right),\tau\right)$ and since $C^\infty\left(\mathbb{T}^n_\Theta\right)\approx\mathcal{S}\left(\mathbb{Z}^n\right)$ there is a linear map $\Psi_\Theta:\mathcal{S}\left(\mathbb{Z}^n\right)\to L^2\left(C\left(\mathbb{T}^n_\Theta\right),\tau\right)$. If $k\in\mathbb{Z}^n$ and $U_k\in\mathcal{S}\left(\mathbb{Z}^n\right)=C^\infty\left(\mathbb{T}^n_\Theta\right)$ is such that

$$U_k\left(p\right)=\delta_{kp}:\ \forall p\in\mathbb{Z}^n \tag{5.10}$$

then

$$U_kU_p=e^{-\pi ik\,\cdot\,\Theta p}U_{k+p};\quad U_kU_p=e^{-2\pi ik\,\cdot\,\Theta p}U_pU_k. \tag{5.11}$$

If $\xi_k=\Psi_\Theta\left(U_k\right)$ then from (5.5), (5.6) it turns out

$$\tau\left(U_k^*\star_\Theta U_l\right)=\left(\xi_k,\xi_l\right)=\delta_{kl}, \tag{5.12}$$

i.e. the subset $\left\{\xi_k\right\}_{k\in\mathbb{Z}^n}\subset L^2\left(C\left(\mathbb{T}^n_\Theta\right),\tau\right)$ is an orthogonal basis of $L^2\left(C\left(\mathbb{T}^n_\Theta\right),\tau\right)$. Hence the Hilbert space $L^2\left(C\left(\mathbb{T}^n_\Theta\right),\tau\right)$ is naturally isomorphic to the Hilbert space $\ell^2\left(\mathbb{Z}^n\right)$ given by

$$\ell^2\left(\mathbb{Z}^n\right)=\left\{\xi=\left\{\xi_k\in\mathbb{C}\right\}_{k\in\mathbb{Z}^n}\in\mathbb{C}^{\mathbb{Z}^n}\mid\sum_{k\in\mathbb{Z}^n}\left|\xi_k\right|^2<\infty\right\}$$

and the $\mathbb{C}$-valued scalar product on $\ell^2\left(\mathbb{Z}^n\right)$ is given by

$$\left(\xi,\eta\right)_{\ell^2\left(\mathbb{Z}^n\right)}=\sum_{k\in\mathbb{Z}^n}\overline{\xi}_k\eta_k.$$

The map $\Psi_\Theta:\mathcal{S}\left(\mathbb{Z}^n\right)\to L^2\left(C\left(\mathbb{T}^n_\Theta\right),\tau\right)$ can be extended up to the map

$$\Psi_\Theta:C\left(\mathbb{T}^n_\Theta\right)\to L^2\left(C\left(\mathbb{T}^n_\Theta\right),\tau\right). \tag{5.13}$$

From (5.8) it follows that for any $a,b\in C^\infty\left(\mathbb{T}^n_\Theta\right)$ the scalar product on $L^2\left(C\left(\mathbb{T}^n_\Theta\right),\tau\right)$ is given by

$$\left(a,b\right)=\int_{\mathbb{T}^n}a^*_{\text{comm}}b_{\text{comm}}dx \tag{5.14}$$

where $a_{\text{comm}}\in C^\infty\left(\mathbb{T}^n\right)$ (resp. b_{comm}) is a commutative function which corresponds to a (resp. b). An alternative description of $\mathbb{C}\left(\mathbb{T}^n_\Theta\right)$ is such that if

$$\Theta=\begin{pmatrix}0&\theta_{12}&\dots&\theta_{1n}\\\theta_{21}&0&\dots&\theta_{2n}\\\vdots&\vdots&\ddots&\vdots\\\theta_{n1}&\theta_{n2}&\dots&0\end{pmatrix} \tag{5.15}$$

then $C\left(\mathbb{T}^n_\Theta\right)$ is the universal C^*-algebra generated by unitary elements $u_1, ..., u_n \in U\left(C\left(\mathbb{T}^n_\Theta\right)\right)$ such that following condition holds

$$u_j u_k = e^{-2\pi i \theta_{jk}} u_k u_j. \tag{5.16}$$

Unitary operators $u_1, ..., u_n$ correspond to the standard basis of $\mathbb{Z}^n$.

Definition 5.2. Unitary elements $u_1, ..., u_n \in U\left(C\left(\mathbb{T}^n_\theta\right)\right)$ which satisfy the relation (5.16) are said to be *generators* of $C\left(\mathbb{T}^n_\Theta\right)$. The set $\{U_l\}_{l\in\mathbb{Z}^n}$ is said to be the *basis* of $C\left(\mathbb{T}^n_\Theta\right)$.

Definition 5.3. If Θ is non-degenerated, that is to say, $\sigma(s,t) \stackrel{\text{def}}{=} s \cdot \Theta t$ to be *symplectic*. This implies even dimension, $n = 2N$. One then selects

$$\Theta = \theta J \stackrel{\text{def}}{=} \theta \begin{pmatrix} 0 & 1_N \\ -1_N & 0 \end{pmatrix} \tag{5.17}$$

where $\theta > 0$ is defined by $\theta^{2N} \stackrel{\text{def}}{=} \det \Theta$. Denote by $C^\infty\left(\mathbb{T}^{2N}_\theta\right) \stackrel{\text{def}}{=} C^\infty\left(\mathbb{T}^{2N}_\Theta\right)$ and $C\left(\mathbb{T}^{2N}_\theta\right) \stackrel{\text{def}}{=} C\left(\mathbb{T}^{2N}_\Theta\right)$.

5.3 Geometry of noncommutative tori

Denote by δ_μ ($\mu = 1, \ldots, n$) the analogues of the partial derivatives $\frac{1}{i}\frac{\partial}{\partial x^\mu}$ on $C^\infty(\mathbb{T}^n)$ which are derivations on the algebra $C^\infty(\mathbb{T}^n_\Theta)$ given by

$$\delta_\mu(U_k) = k_\mu U_k.$$

These derivations have the following property

$$\delta_\mu(a^*) = -(\delta_\mu a)^*,$$

and also satisfy the integration by parts formula

$$\tau(a\delta_\mu b) = -\tau((\delta_\mu a)b), \quad a, b \in C^\infty(\mathbb{T}^n_\Theta).$$

The spectral triple describing the noncommutative geometry of noncommutative n-torus consists of the algebra $C^\infty(\mathbb{T}^n_\Theta)$, the Hilbert space $\mathcal{H} = L^2\left(C\left(\mathbb{T}^n_\Theta\right), \tau\right) \otimes \mathbb{C}^m$, where $m = 2^{[n/2]}$ with the representation of $C^\infty(\mathbb{T}^n_\Theta)$ given by $\pi \otimes 1_{B\left(\mathbb{C}^N\right)}$. The Dirac operator is given by

$$D = \partial\!\!\!/ \stackrel{\text{def}}{=} \sum_{\mu=1}^{n} \partial_\mu \otimes \gamma^\mu \cong \sum_{\mu=1}^{n} \delta_\mu \otimes \gamma^\mu, \tag{5.18}$$

where $\partial_\mu = \delta_\mu$, seen as an unbounded self-adjoint operator on $\mathcal{H}$ and γ^μs are Clifford (Gamma) matrices in $M_N(\mathbb{C})$ satisfying the relation

$$\gamma^i\gamma^j + \gamma^j\gamma^i = 2\delta^{ij} I_N.$$

There is a spectral triple

$$\left(C^\infty(\mathbb{T}^n_\Theta), L^2\left(C\left(\mathbb{T}^n_\Theta\right), \tau\right) \otimes \mathbb{C}^m, D\right). \tag{5.19}$$

There is an alternative description of D. The space $C^\infty\left(\mathbb{T}^n\right)$ (resp. $C^\infty\left(\mathbb{T}^n_\Theta\right)$) is dense in $L^2\left(\mathbb{T}^n\right)$ (resp. $L^2\left(C^\infty\left(\mathbb{T}^n_\Theta\right), \tau\right)$), hence from the $\mathbb{C}$-linear isomorphism $\varphi_\infty : C^\infty\left(\mathbb{T}^n_\Theta\right) \xrightarrow{\approx} C^\infty\left(\mathbb{T}^n\right)$ given by (5.7) it follows isomorphism of Hilbert spaces

$$\varphi : L^2\left(C\left(\mathbb{T}^n_\Theta\right), \tau\right) \xrightarrow{\approx} L^2\left(\mathbb{T}^n\right).$$

Otherwise $\mathbb{T}^n$ admits a Spin-bundle S such that $L^2\left(\mathbb{T}^2, S\right) \approx L^2\left(\mathbb{T}^n\right) \otimes \mathbb{C}^m$. It turns out an isomorphism of Hilbert spaces

$$\mathit{\Phi} : L^2\left(C\left(\mathbb{T}^n_\Theta\right), \tau\right) \otimes \mathbb{C}^m \xrightarrow{\approx} L^2\left(\mathbb{T}^n, S\right).$$

There is a commutative spectral triple

$$\left(C^\infty\left(\mathbb{T}^n\right), L^2\left(\mathbb{T}^n, S\right), \not{D}\right) \tag{5.20}$$

such that D is given by

$$D = \Phi^{-1} \circ \not{D} \circ \Phi. \tag{5.21}$$

Noncommutative geometry replaces differentials with commutators such that the differential df corresponds to $\frac{1}{i}\left[\not{D}, f\right]$ and the well known equation

$$df = \sum_{\mu=1}^{n} \frac{\partial f}{\partial x_\mu} dx_\mu$$

is replaced with

$$\left[\not{D}, f\right] = \sum_{\mu=1}^{n} \frac{\partial f}{\partial x_\mu}\left[\not{D}, x_\mu\right] \tag{5.22}$$

In case of commutative torus we on has

$$dx_\mu = i u^*_\mu du_\mu$$

where $u_\mu = e^{-ix_\mu}$, so what equation (5.22) can be written by the following way

$$\left[\not{D}, f\right] = \sum_{\mu=1}^{n} \frac{\partial f}{\partial x_\mu} u^*_\mu \left[\not{D}, u_\mu\right] \tag{5.23}$$

We would like to prove a noncommutative analog of (5.23), i.e. for any $a \in C^\infty\left(\mathbb{T}^n_\Theta\right)$ following condition holds

$$\left[D, a\right] = \sum_{\mu=1}^{n} \frac{\partial a}{\partial x_\mu} u^*_\mu \left[D, u_\mu\right] \tag{5.24}$$

From (5.18) it follows that (5.24) is true if and only if

$$\left[\delta_\mu, a\right] = \sum_{\mu=1}^{n} \frac{\partial a}{\partial x_\mu} u_\mu^* \left[\delta_\mu, u_\mu\right];\ \mu = 1, \dots n. \tag{5.25}$$

In the above equation $\frac{\partial a}{\partial x_\mu}$ means that one considers a as element of $C^\infty\left(\mathbb{T}^n\right)$, takes $\frac{\partial}{\partial x_\mu}$ of it and then the result of derivation considers as element of $C^\infty\left(\mathbb{T}^n_\Theta\right)$. Since the linear span of elements U_k is dense in both $C^\infty\left(\mathbb{T}^n_\Theta\right)$ and $L^2\left(C\left(\mathbb{T}^n_\Theta\right), \tau\right)$ the equation (5.25) is true if for any $k, l \in \mathbb{Z}^n$ following condition holds

$$\left[\delta_\mu, U_k\right] U_l = \frac{\partial U_k}{\partial x_\mu} u_\mu^* \left[\delta_\mu, u_\mu\right] U_l.$$

The above equation is a consequence of the following calculations:

$$\left[\delta_\mu, U_k\right] U_l = \delta_\mu U_k U_l - U_k \delta_k U_l = (k+l) U_k U_l - l U_k U_l = k U_k U_l,$$

$$\frac{\partial U_k}{\partial x_\mu} u_\mu^* \left[\delta_\mu, u_\mu\right] U_l = k U_k u^* \left(\delta_\mu u_\mu U_l - u_\mu \delta U_l\right) = k U_k u^* \left(\left(l+1\right) u_\mu U_l - l u_\mu \delta U_l\right) =$$

$$= k U_k u_\mu^* u_\mu U_l = k U_k U_l.$$

For any $k \in \mathbb{Z}^n$ following condition holds

$$u_\mu^* \left[\delta_\mu, u_\mu\right] U_k = u_\mu^* \delta_\mu u_\mu U_k - u_\mu^*\ u_\mu \delta_\mu U_k = u_\mu^* \left(\left(k+1\right) u_\mu U_k - k u_\mu U_k\right) = U_k$$

it turns out

$$u_\mu^* \left[\delta_\mu, u_\mu\right] = 1_{C\left(\mathbb{T}^n_\Theta\right)} \tag{5.26}$$

5.4 Finite-fold coverings

5.4.1 Basic construction

In this section we write ab instead $a \star_\Theta b$. Let Θ be given by (5.15), and let $C\left(\mathbb{T}^n_\Theta\right)$ be a noncommutative torus. If $\overline{k} = (k_1, ..., k_n) \in \mathbb{N}^n$ and

$$\widetilde{\Theta} = \begin{pmatrix} 0 & \widetilde{\theta}_{12} & \dots & \widetilde{\theta}_{1n} \\ \widetilde{\theta}_{21} & 0 & \dots & \widetilde{\theta}_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \widetilde{\theta}_{n1} & \widetilde{\theta}_{n2} & \dots & 0 \end{pmatrix}$$

is a skew-symmetric matrix such that

$$e^{-2\pi i \theta_{rs}} = e^{-2\pi i \widetilde{\theta}_{rs} k_r k_s}$$

then there is a *-homomorphism $C\left(\mathbb{T}^n_\Theta\right) \to C\left(\mathbb{T}^n_{\widetilde{\Theta}}\right)$ given by

$$u_j \mapsto v_j^{k_j};\ j = 1, ..., n \tag{5.27}$$

where $u_1, ..., u_n \in C\left(\mathbb{T}^n_\Theta\right)$ (resp. $v_1, ..., v_n \in C\left(\mathbb{T}^n_{\widetilde{\Theta}}\right)$) are unitary generators of $C\left(\mathbb{T}^n_\Theta\right)$ (resp. $C\left(\mathbb{T}^n_{\widetilde{\Theta}}\right)$). There is an involutive action of $G = \mathbb{Z}_{k_1} \times ... \times \mathbb{Z}_{k_n}$ on $C\left(\mathbb{T}^n_{\widetilde{\Theta}}\right)$ given by

$$\left(\overline{p}_1, ..., \overline{p}_n\right) v_j = e^{\frac{2\pi i p_j}{k_j}} v_j,$$

and a following condition holds $C\left(\mathbb{T}^n_\Theta\right) = C\left(\mathbb{T}^n_{\widetilde{\Theta}}\right)^G$. Otherwise there is a following $C\left(\mathbb{T}^n_\Theta\right)$ - module isomorphism

$$C\left(\mathbb{T}^n_{\widetilde{\Theta}}\right) = \bigoplus_{\left(\overline{p}_1, ...\overline{p}_n\right) \in \mathbb{Z}_{k_1} \times ... \times \mathbb{Z}_{k_n}} v_1^{p_1} \cdot ... \cdot v_n^{p_n} C\left(\mathbb{T}^n_\Theta\right) \approx C\left(\mathbb{T}^n_\Theta\right)^{k_1 \cdot ... \cdot k_n}$$

i.e. $C\left(\mathbb{T}^n_{\widetilde{\Theta}}\right)$ is a finitely generated projective Hilbert $C\left(\mathbb{T}^n_\Theta\right)$-module. It turns out the following theorem.

Theorem 5.4. *[12] The triple* $\left(C\left(\mathbb{T}^n_\Theta\right), C\left(\mathbb{T}^n_{\widetilde{\Theta}}\right), \mathbb{Z}_{k_1} \times ... \times \mathbb{Z}_{k_n}\right)$ *is an unital noncommutative finite-fold covering.*

5.4.2 Induced representation

Similarly to (2.6) we consider following pre-Hilbert space

$$C\left(\mathbb{T}^n_{\widetilde{\Theta}}\right) \otimes_{C\left(\mathbb{T}^n_\Theta\right)} L^2\left(C\left(\mathbb{T}^n_\Theta\right), \tau\right)$$

and denote by $\widetilde{\mathcal{H}}$ its Hilbert completion. There are dense subspaces $C^\infty\left(\mathbb{T}^n_{\widetilde{\Theta}}\right) \subset C\left(\mathbb{T}^n_{\widetilde{\Theta}}\right)$, $C^\infty\left(\mathbb{T}^n_\Theta\right) \subset L^2\left(C\left(\mathbb{T}^n_\Theta\right), \tau\right)$, hence the composition

$$C^\infty\left(\mathbb{T}^n_{\widetilde{\Theta}}\right) \otimes_{C^\infty\left(\mathbb{T}^n_\Theta\right)} C^\infty\left(\mathbb{T}^n_\Theta\right) \subset C\left(\mathbb{T}^n_{\widetilde{\Theta}}\right) \otimes_{C\left(\mathbb{T}^n_\Theta\right)} L^2\left(C\left(\mathbb{T}^n_\Theta\right), \tau\right) \subset \widetilde{\mathcal{H}}$$

is the dense inclusion. Otherwise $C^\infty\left(\mathbb{T}^n_{\widetilde{\Theta}}\right) \otimes_{C^\infty\left(\mathbb{T}^n_\Theta\right)} C^\infty\left(\mathbb{T}^n_\Theta\right) \cong C^\infty\left(\mathbb{T}^n_{\widetilde{\Theta}}\right)$ it follows that there is the dense (with respect to the topology of the Hilbert space) inclusion

$$C^\infty\left(\mathbb{T}^n_{\widetilde{\Theta}}\right) \subset \widetilde{\mathcal{H}}.$$

If $\widetilde{a}, \widetilde{b} \in C^\infty\left(\mathbb{T}^n_{\widetilde{\Theta}}\right)$ then from (2.6) it turns out

$$\left(\widetilde{a} \otimes \Psi_\Theta\left(1_{C\left(\mathbb{T}^n_\Theta\right)}\right), \widetilde{b} \otimes \Psi_\Theta\left(1_{C\left(\mathbb{T}^n_\Theta\right)}\right)\right)_{\widetilde{\mathcal{H}}} =$$

$$= \left(\Psi_\Theta\left(1_{C\left(\mathbb{T}^n_\Theta\right)}\right), \left\langle \widetilde{a}, \widetilde{b} \right\rangle_{C\left(\mathbb{T}^n_\Theta\right)} \Psi_\Theta\left(1_{C\left(\mathbb{T}^n_\Theta\right)}\right)\right)_{L^2\left(C\left(\mathbb{T}^n_\Theta\right), \tau\right)} =$$

$$= \int_{\mathbb{T}^n} \left(\sum_{g \in \mathbb{Z}_{k_1} \times ... \times \mathbb{Z}_{k_n}} g\left(\widetilde{a}^*_{\text{comm}} \widetilde{b}_{\text{comm}}\right)\right)(x)\ dx = \int_{\widetilde{\mathbb{T}}^n} \widetilde{a}^*_{\text{comm}} \widetilde{b}_{\text{comm}}\left(\widetilde{x}\right)\ d\widetilde{x}$$

where $\widetilde{a}_{\text{comm}} \in C^\infty(\mathbb{T}^n)$ (resp. $\widetilde{a}_{\text{comm}}$) is a commutative function which corresponds to $\widetilde{a}$ (resp. $\widetilde{a}$). Above formula coincides with (5.14). Taking into account that $C^\infty\left(\mathbb{T}^n_{\widetilde{\Theta}}\right)$ is dense in the Hilbert space $L^2\left(C\left(\mathbb{T}^n_{\widetilde{\Theta}}\right), \widetilde{\tau}\right)$ one has an isomorphism

$$\widetilde{\mathcal{H}} \approx L^2\left(C\left(\mathbb{T}^n_{\widetilde{\Theta}}\right), \widetilde{\tau}\right) \quad (5.28)$$

of Hilbert spaces. Thus if $\rho : C\left(\mathbb{T}^n_{\Theta}\right) \to L^2\left(C\left(\mathbb{T}^n_{\Theta}\right), \tau\right)$ then $\widetilde{\rho} : C\left(\mathbb{T}^n_{\widetilde{\Theta}}\right) \to B\left(L^2\left(C\left(\mathbb{T}^n_{\widetilde{\Theta}}\right), \widetilde{\tau}\right)\right)$ is induced by the pair $\left(\rho, \left(C\left(\mathbb{T}^n_{\Theta}\right), C\left(\mathbb{T}^n_{\widetilde{\Theta}}\right), G\left(C\left(\mathbb{T}^n_{\widetilde{\Theta}}\right) \mid C\left(\mathbb{T}^n_{\Theta}\right)\right)\right)\right)$.

5.4.3 Coverings of spectral triples

Let us consider following objects

- The spectral triple $\left(C^\infty(\mathbb{T}^n_{\Theta}), L^2\left(C\left(\mathbb{T}^n_{\Theta}\right), \tau\right) \otimes \mathbb{C}^m, D\right)$ given by (5.19),
- An unital noncommutative finite-fold covering $\left(C\left(\mathbb{T}^n_{\Theta}\right), C\left(\mathbb{T}^n_{\widetilde{\Theta}}\right), \mathbb{Z}_{k_1} \times ... \times \mathbb{Z}_{k_n}\right)$ given by the Theorem 5.4.

Let $\varphi_\infty : C^\infty\left(\mathbb{T}^n_{\Theta}\right) \xrightarrow{\approx} C^\infty\left(\mathbb{T}^n\right)$ be a $\mathbb{C}$-linear isomorphism given by (5.7) and suppose that $x_1, ..., x_n \in C\left(\mathbb{T}^n\right)$ are unitary generators of $C\left(\mathbb{T}^n\right)$. Let $\pi_{\text{comm}} : C\left(\mathbb{T}^n\right) \to C\left(\widetilde{\mathbb{T}^n}\right)$ be a *-homomorphism which corresponds to a finite-fold covering of commutative torus. Clearly $\widetilde{\mathbb{T}^n} \approx \mathbb{T}^n$. We suppose that π_{comm} is given by

$$x_j \mapsto y_j^{k_j}.$$

where $y_1, ..., y_n \in C\left(\widetilde{\mathbb{T}}^n\right)$ are unitary generators of $C\left(\widetilde{\mathbb{T}}^n\right)$. There is a topological covering $\varphi : \widetilde{\mathbb{T}^n} \to \mathbb{T}^n$ induced by *-homomorphism π_{comm}. There is a commutative spectral triple $\left(C^\infty\left(\mathbb{T}^n\right), L^2\left(\mathbb{T}^n, S\right), \not{D}\right)$ given by (5.20). Denote by $\widetilde{S} = \varphi^* S$, $\widetilde{\not{D}} = \varphi^* \not{D}$ inverse images of the Spin-bundle S and Dirac operator $\not{D}$ (cf. 1.2.2, 1.13). From (5.28) it turns out that the representation $\widetilde{\rho} : C\left(\mathbb{T}^n_{\widetilde{\Theta}}\right) \to B\left(L^2\left(C\left(\mathbb{T}^n_{\widetilde{\Theta}}\right), \widetilde{\tau}\right) \otimes \mathbb{C}^m\right)$ is induced by

$$\left(\rho, \left(C\left(\mathbb{T}^n_{\Theta}\right), C\left(\mathbb{T}^n_{\widetilde{\Theta}}\right), \mathbb{Z}_{k_1} \times ... \times \mathbb{Z}_{k_n}\right)\right),$$

where $\rho : C\left(\mathbb{T}^n_{\Theta}\right) \to B\left(L^2\left(C\left(\mathbb{T}^n_{\Theta}\right), \tau\right) \otimes \mathbb{C}^m\right)$. Otherwise there is a natural $\mathbb{C}$-linear isomorphism

$$\widetilde{\varphi} : L^2\left(C\left(\mathbb{T}^n_{\widetilde{\Theta}}\right) \otimes \mathbb{C}^m, \tau\right) \xrightarrow{\approx} L^2\left(\widetilde{\mathbb{T}}^n, \widetilde{S}\right)$$

of Hilbert spaces. Denote by

$$\widetilde{D} = \widetilde{\varphi}^{-1} \circ \widetilde{\not{D}} \circ \widetilde{\varphi}.$$

Clearly $C^\infty\left(\mathbb{T}^n_{\widetilde{\Theta}}\right)$ is dense in $C\left(\mathbb{T}^n_{\widetilde{\Theta}}\right)$ and operator $\widetilde{D}$ satisfies to conditions (b), (c) of the Definition 2.14. In result one has a following theorem.

Theorem 5.5. *The triple* $\left(C^\infty\left(\mathbb{T}^n_{\widetilde{\Theta}}\right), L^2\left(C\left(\mathbb{T}^n_{\widetilde{\Theta}}\right)\right), \widetilde{D}\right)$ *is a* $\left(C\left(\mathbb{T}^n_{\Theta}\right), C\left(\mathbb{T}^n_{\widetilde{\Theta}}\right), \mathbb{Z}_{k_1}\times\ldots\times\mathbb{Z}_{k_n}\right)$*-lift of* $\left(C^\infty(\mathbb{T}^n_\Theta), L^2\left(C\left(\mathbb{T}^n_\Theta\right), \tau\right)\otimes\mathbb{C}^m, D\right)$.

Similarly to (5.24) for any $\widetilde{a}\in C^\infty\left(\mathbb{T}^n_{\widetilde{\Theta}}\right)$ following condition holds

$$\left[\widetilde{D}, \widetilde{a}\right] = \sum_{\mu=1}^{n}\frac{\partial\widetilde{a}}{\partial x_\mu}u^*_\mu\left[D, u_\mu\right] \tag{5.29}$$

where $u_1, \ldots, u_n$ are unitary generators of $C^\infty\left(\mathbb{T}^n_\Theta\right)$.

5.5 Moyal plane and a representation of the noncommutative torus

Definition 5.6. Denote the *Moyal plane* product $\star_\theta$ on $\mathcal{S}\left(\mathbb{R}^{2N}\right)$ given by

$$\left(f\star_\theta h\right)(u) = \int_{y\in\mathbb{R}^{2N}} f\left(u-\frac{1}{2}\Theta y\right)g\left(u+v\right)e^{2\pi i y\cdot v}dydv \tag{5.30}$$

where Θ is given by (5.17).

Definition 5.7. [7] Denote by $\mathcal{S}'\left(\mathbb{R}^n\right)$ the vector space dual to $\mathcal{S}\left(\mathbb{R}^n\right)$, i.e. the space of continuous functionals on $\mathcal{S}\left(\mathbb{R}^n\right)$. The Moyal product can be defined, by duality, on larger sets than $\mathcal{S}\left(\mathbb{R}^{2N}\right)$. For $T\in\mathcal{S}'\left(\mathbb{R}^{2N}\right)$, write the evaluation on $g\in\mathcal{S}\left(\mathbb{R}^{2N}\right)$ as $\langle T, g\rangle\in\mathbb{C}$; then, for $f\in\mathcal{S}$ we may define $T\star_\theta f$ and $f\star_\theta T$ as elements of $\mathcal{S}'\left(\mathbb{R}^{2N}\right)$ by

$$\begin{aligned}\langle T\star_\theta f, g\rangle &\stackrel{\text{def}}{=} \langle T, f\star_\theta g\rangle\\ \langle f\star_\theta T, g\rangle &\stackrel{\text{def}}{=} \langle T, g\star_\theta f\rangle\end{aligned} \tag{5.31}$$

using the continuity of the star product on $\mathcal{S}\left(\mathbb{R}^{2N}\right)$. Also, the involution is extended to by $\langle T^*, g\rangle \stackrel{\text{def}}{=} \overline{\langle T, g^*\rangle}$. Consider the left and right multiplier algebras:

$$\begin{aligned}&\mathcal{M}^\theta_L \stackrel{\text{def}}{=} \{\, T\in\mathcal{S}'(\mathbb{R}^{2N}) : T\star_\theta h\in\mathcal{S}(\mathbb{R}^{2N}) \text{ for all } h\in\mathcal{S}(\mathbb{R}^{2N})\,\},\\ &\mathcal{M}^\theta_R \stackrel{\text{def}}{=} \{\, T\in\mathcal{S}'(\mathbb{R}^{2N}) : h\star_\theta T\in\mathcal{S}(\mathbb{R}^{2N}) \text{ for all } h\in\mathcal{S}(\mathbb{R}^{2N})\,\},\\ &\mathcal{M}^\theta \stackrel{\text{def}}{=} \mathcal{M}^\theta_L\cap\mathcal{M}^\theta_R.\end{aligned} \tag{5.32}$$

In [7] it is proven that

$$\mathcal{M}^\theta_R\star_\theta\mathcal{S}'\left(\mathbb{R}^{2N}\right) = \mathcal{S}'\left(\mathbb{R}^{2N}\right) \text{ and } \mathcal{S}'\left(\mathbb{R}^{2N}\right)\star_\theta\mathcal{M}^\theta_L = \mathcal{S}'\left(\mathbb{R}^{2N}\right). \tag{5.33}$$

It is known [10] that the domain of the Moyal plane product can be extended up to $L^2\left(\mathbb{R}^{2N}\right)$.

Lemma 5.8. *[10] If $f, g \in L^2\left(\mathbb{R}^{2N}\right)$, then $f \star_\theta g \in L^2\left(\mathbb{R}^{2N}\right)$ and $\|f\|_{\mathrm{op}} < (2\pi\theta)^{-\frac{N}{2}} \|f\|_2$. where $\|\cdot\|_2$ be the L^2-norm given by*

$$\|f\|_2 \stackrel{\mathrm{def}}{=} \left| \int_{\mathbb{R}^{2N}} |f|^2 \, dx \right|^{\frac{1}{2}}. \tag{5.34}$$

and the operator norm

$$\|T\|_{\mathrm{op}} \stackrel{\mathrm{def}}{=} \sup\{ \|T \star g\|_2 / \|g\|_2 : 0 \neq g \in L^2\left(\left(\mathbb{R}^{2N}\right)\right) \} \tag{5.35}$$

Definition 5.9. Denote by $\mathcal{S}\left(\mathbb{R}^{2N}_\theta\right)$ (resp. $L^2\left(\mathbb{R}^{2N}_\theta\right)$) the operator algebra which is $\mathbb{C}$-linearly isomorphic to $\mathcal{S}\left(\mathbb{R}^{2N}\right)$ (resp. $L^2\left(\mathbb{R}^{2N}\right)$) and product coincides with $\star_\theta$. Both $\mathcal{S}\left(\mathbb{R}^{2N}_\theta\right)$ and $L^2\left(\mathbb{R}^{2N}_\theta\right)$ act on the Hilbert space $L^2\left(\mathbb{R}^{2N}\right)$. Denote by

$$\Psi_\theta : \mathcal{S}\left(\mathbb{R}^{2N}\right) \xrightarrow{\approx} \mathcal{S}\left(\mathbb{R}^{2N}_\theta\right) \tag{5.36}$$

the natural $\mathbb{C}$-linear isomorphism.

5.10. There is the tracial property [10] of the Moyal product

$$\int_{\mathbb{R}^{2N}} (f \star_\theta g)(x) \, dx = \int_{\mathbb{R}^{2N}} f(x) \, g(x) \, dx. \tag{5.37}$$

The Fourier transformation of the star product satisfies to the following condition.

$$\mathcal{F}(f \star_\theta g)(x) = \int_{\mathbb{R}^{2N}} \mathcal{F}f(x-y) \, \mathcal{F}g(y) \, e^{\pi i y \cdot \Theta x} \, dy. \tag{5.38}$$

Definition 5.11. [10] Let $\mathcal{S}'\left(\mathbb{R}^{2N}\right)$ be a vector space dual to $\mathcal{S}\left(\mathbb{R}^{2N}\right)$. Denote by $C_b\left(\mathbb{R}^{2N}_\theta\right) \stackrel{\mathrm{def}}{=} \{ T \in \mathcal{S}'\left(\mathbb{R}^{2N}\right) : T \star_\theta g \in L^2\left(\mathbb{R}^{2N}\right) \text{ for all } g \in L^2(\mathbb{R}^{2N}) \}$, provided with the operator norm

$$\|T\|_{\mathrm{op}} \stackrel{\mathrm{def}}{=} \sup\{ \|T \star_\theta g\|_2 / \|g\|_2 : 0 \neq g \in L^2(\mathbb{R}^{2N}) \}. \tag{5.39}$$

Denote by $C_0\left(\mathbb{R}^{2N}_\theta\right)$ the operator norm completion of $\mathcal{S}\left(\mathbb{R}^{2N}_\theta\right)$.

Remark 5.12. Obviously $\mathcal{S}\left(\mathbb{R}^{2N}_\theta\right) \hookrightarrow C_b\left(\mathbb{R}^{2N}_\theta\right)$. But $\mathcal{S}\left(\mathbb{R}^{2N}_\theta\right)$ is not dense in $C_b\left(\mathbb{R}^{2N}_\theta\right)$, i.e. $C_0\left(\mathbb{R}^{2N}_\theta\right) \subsetneq C_b\left(\mathbb{R}^{2N}_\theta\right)$ (cf. [10]).

Remark 5.13. $L^2\left(\mathbb{R}^{2N}_\theta\right)$ is the $\|\cdot\|_2$ norm completion of $\mathcal{S}\left(\mathbb{R}^{2N}_\theta\right)$ hence from the Lemma 5.8 it follows that

$$L^2\left(\mathbb{R}^{2N}_\theta\right) \subset C_0\left(\mathbb{R}^{2N}_\theta\right). \tag{5.40}$$

Remark 5.14. Notation of the Definition 5.11 differs from [10]. Here symbols $A_\theta, \mathcal{A}_\theta, A^0_\theta$ are replaced with $C_b\left(\mathbb{R}^{2N}_\theta\right), \mathcal{S}\left(\mathbb{R}^{2N}_\theta\right), C_0\left(\mathbb{R}^{2N}_\theta\right)$ respectively.

Remark 5.15. The $\mathbb{C}$-linear space $C_0\left(\mathbb{R}^{2N}_\theta\right)$ is not isomorphic to $C_0\left(\mathbb{R}^{2N}\right)$.

There are elements $\left\{f_{nm} \in \mathcal{S}\left(\mathbb{R}^2\right)\right\}_{m,n\in\mathbb{N}^0}$, described in [7], which satisfy to the following proposition.

Proposition 5.16. *[7,10] Let $N = 1$. Then $\mathcal{S}\left(\mathbb{R}^{2N}_\theta\right) = \mathcal{S}\left(\mathbb{R}^2_\theta\right)$ has a Fréchet algebra isomorphism with the matrix algebra of rapidly decreasing double sequences $c = (c_{mn})$ such that, for each $k \in \mathbb{N}$,*

$$r_k(c) \stackrel{\mathrm{def}}{=} \left(\sum_{m,n=0}^{\infty} \theta^{2k}\left(m+\tfrac{1}{2}\right)^k\left(n+\tfrac{1}{2}\right)^k |c_{mn}|^2\right)^{1/2} \tag{5.41}$$

is finite, topologized by all the seminorms (r_k); via the decomposition $f = \sum_{m,n=0}^{\infty} c_{mn} f_{mn}$ of $\mathcal{S}(\mathbb{R}^2)$ in the $\{f_{mn}\}$ basis. The twisted product $f \star_\theta g$ is the matrix product ab, where

$$(ab)_{mn} \stackrel{\mathrm{def}}{=} \sum_{k=0}^{\infty} a_{mk} b_{kn}. \tag{5.42}$$

For $N > 1$, $C^\infty\left(\mathbb{R}^{2N}_\theta\right)$ is isomorphic to the (projective) tensor product of N matrix algebras of this kind, i.e.

$$\mathcal{S}\left(\mathbb{R}^{2N}_\theta\right) \cong \underbrace{\mathcal{S}\left(\mathbb{R}^2_\theta\right) \otimes \cdots \otimes \mathcal{S}\left(\mathbb{R}^2_\theta\right)}_{N-\text{times}} \tag{5.43}$$

with the projective topology induced by seminorms r_k given by (5.41).

Remark 5.17. If A is C^*-norm completion of the matrix algebra with the norm (5.41) then $A \approx \mathcal{K}$, i.e.

$$C_0\left(\mathbb{R}^2_\theta\right) \approx \mathcal{K}. \tag{5.44}$$

Form (5.43) and (5.44) it follows that

$$C_0\left(\mathbb{R}^{2N}_\theta\right) \cong \underbrace{C_0\left(\mathbb{R}^2_\theta\right) \otimes \cdots \otimes C_0\left(\mathbb{R}^2_\theta\right)}_{N-\text{times}} \approx \underbrace{\mathcal{K} \otimes \cdots \otimes \mathcal{K}}_{N-\text{times}} \approx \mathcal{K} \tag{5.45}$$

where $\otimes$ means minimal or maximal tensor product ($\mathcal{K}$ is nuclear hence both products coincide).

5.18. [10] By plane waves we understand all functions of the form

$$x \mapsto \exp(ik \cdot x)$$

for $k \in \mathbb{R}^{2N}$. One obtains for the Moyal product of plane waves:

$$\exp(ik\cdot) \star_\Theta \exp(ik\cdot) = \exp(ik\cdot) \star_\theta \exp(ik\cdot) = \exp(i(k+l)\cdot)\, e^{-\pi i k\cdot\Theta l} \tag{5.46}$$

5.19. The equation (5.46) is similar to the equation (5.11) which defines $C\left(\mathbb{T}^n_\Theta\right)$. This fact enables us to construct a representation $\pi : C\left(\mathbb{T}^n_\Theta\right) \to B\left(L^2\left(\mathbb{R}^{2N}\right)\right)$

$$\begin{aligned} \pi : C\left(\mathbb{T}^n_\Theta\right) &\to B\left(L^2\left(\mathbb{R}^{2N}\right)\right), \\ U_k &\mapsto \exp(2\pi i k\cdot) \end{aligned} \tag{5.47}$$

where $U_k \in C\left(\mathbb{T}^n_\Theta\right)$ is given by the Definition 5.2.

5.20. Let us consider the unitary dilation operators E_a given by

$$E_a f(x) \stackrel{\text{def}}{=} a^{N/2} f(a^{1/2}x),$$

It is proven in [10] that

$$f \star_\theta g = (\theta/2)^{-N/2} E_{2/\theta}(E_{\theta/2} f \star_2 E_{\theta/2} g). \tag{5.48}$$

We can simplify our construction by setting $\theta = 2$. Thanks to the scaling relation (5.48) any qualitative result can is true if it is true in case of $\theta = 2$. We use the following notation

$$f \times g \stackrel{\text{def}}{=} f \star_2 g \tag{5.49}$$

Definition 5.21. [10] We may as well introduce more Hilbert spaces $\mathcal{G}_{st}$ (for $s, t \in \mathbb{R}$) of those

$$f \in \mathcal{S}'(\mathbb{R}^2) = \sum_{m,n=0}^{\infty} c_{mn} f_{mn}$$

for which the following sum is finite:

$$\|f\|_{st}^2 \stackrel{\text{def}}{=} \sum_{m,n=0}^{\infty} (m + \tfrac{1}{2})^s (n + \tfrac{1}{2})^t |c_{mn}|^2.$$

for $\mathcal{G}_{st}$.

Remark 5.22. It is proven in [7] $f, g \in L^2\left(\mathbb{R}^2\right)$, then $f \times g \in L^2\left(\mathbb{R}^2\right)$ and $\|f \times g\| \leq \|f\| \, \|g\|$. Moreover, $f \times g$ lies in $C_0\left(\mathbb{R}^2\right)$: the continuity follows by adapting the analogous argument for (ordinary) convolution.

Remark 5.23. It is shown in [7] that

$$\mathcal{S}\left(\mathbb{R}^2\right) = \bigcap_{s,t \in \mathbb{R}} \mathcal{G}_{st}. \tag{5.50}$$

5.24. This part contains a useful equations proven in [7]. There are coordinate functions p, q on $\mathbb{R}^2$ such that for any $f \in \mathcal{S}\left(\mathbb{R}^2\right)$ following conditions hold

$$\begin{split} q \times f = \left(q + i\frac{\partial}{\partial p}\right) f; \quad p \times f = \left(p - i\frac{\partial}{\partial q}\right) f; \\ f \times q = \left(q - i\frac{\partial}{\partial p}\right) f; \quad f \times p = \left(p + i\frac{\partial}{\partial q}\right) f. \end{split} \tag{5.51}$$

From $q \times f, f \times q, p \times f, f \times p \in \mathcal{S}\left(\mathbb{R}^{2N}\right)$ it follows that $p, q \in \mathcal{M}^2$ (cf. (5.32)). From (5.33) it follows that

$$\begin{split} q \times \mathcal{S}'\left(\mathbb{R}^{2N}\right) \subset \mathcal{S}'\left(\mathbb{R}^{2N}\right); \quad p \times \mathcal{S}'\left(\mathbb{R}^{2N}\right) \subset \mathcal{S}'\left(\mathbb{R}^{2N}\right); \\ \mathcal{S}'\left(\mathbb{R}^{2N}\right) \times q \subset \mathcal{S}'\left(\mathbb{R}^{2N}\right); \quad \mathcal{S}'\left(\mathbb{R}^{2N}\right) \times p \subset \mathcal{S}'\left(\mathbb{R}^{2N}\right). \end{split} \tag{5.52}$$

If $f \in \mathcal{S}'\left(\mathbb{R}^2\right)$ then from (5.51) it follows that

$$\frac{\partial}{\partial p} f = -iq \times f + if \times q, \quad \frac{\partial}{\partial q} f = ip \times f - if \times p \tag{5.53}$$

If

$$\begin{gathered} a \stackrel{\text{def}}{=} \frac{q + ip}{\sqrt{2}}, \quad \overline{a} \stackrel{\text{def}}{=} \frac{q - ip}{\sqrt{2}}, \\ \frac{\partial}{\partial a} \stackrel{\text{def}}{=} \frac{\partial_q + i\partial_p}{\sqrt{2}}, \quad \frac{\partial}{\partial \overline{a}} \stackrel{\text{def}}{=} \frac{\partial_q - i\partial_p}{\sqrt{2}}, \\ H \stackrel{\text{def}}{=} a\overline{a} = \frac{1}{2}\left(p^2 + q^2\right), \\ \overline{a} \times a = H - 1, \quad a \times \overline{a} = H + 1 \end{gathered} \tag{5.54}$$

then

$$\begin{gathered} a \times f = af + \frac{\partial f}{\partial \overline{a}}, \quad f \times a = af - \frac{\partial f}{\partial \overline{a}}, \\ \overline{a} \times f = \overline{a} f - \frac{\partial f}{\partial a}, \quad f \times a = \overline{a} f + \frac{\partial f}{\partial a}, \end{gathered} \tag{5.55}$$

$$H \times f_{mn} = (2m+1) f_{mn}; \quad f_{mn} \times H = 2(n+1) f_{mn} \tag{5.56}$$

$$\begin{gathered} a \times f_{mn} = \sqrt{2m} f_{m-1,n}; \quad f_{mn} \times a = \sqrt{2n+2} f_{m,n+1}; \\ \overline{a} \times f_{mn} = \sqrt{2m+2} f_{m+1,n}; \quad f_{m+1,n} \times \overline{a} = \sqrt{2n} f_{m,n-1}. \end{gathered} \tag{5.57}$$

It is proven in [7] that

$$\partial_j (f \times g) = \partial_j f \times g + f \times \partial_j g; \tag{5.58}$$

where $\partial_j = \frac{\partial}{\partial x_j}$ is the partial derivation in $\mathcal{S}\left(\mathbb{R}^{2N}\right)$.

5.6 Infinite coverings

Let us consider a sequence

$$\mathfrak{S}_{C\left(\mathbb{T}^n_\Theta\right)} = \left\{ C\left(\mathbb{T}^n_\Theta\right) = C\left(\mathbb{T}^n_{\Theta_0}\right) \xrightarrow{\pi^1} ... \xrightarrow{\pi^j} C\left(\mathbb{T}^n_{\Theta_j}\right) \xrightarrow{\pi^{j+1}} ... \right\}. \tag{5.59}$$

of finite coverings of noncommutative tori. The sequence (5.59) satisfies to the Definition 3.1, i.e. $\mathfrak{S}_{C\left(\mathbb{T}^n_\Theta\right)} \in \mathfrak{FinAlg}$.

5.25. Let $\Theta = J\theta$ where $\theta \in \mathbb{R} \backslash \mathbb{Q}$ and

$$J = \begin{pmatrix} 0 & 1_N \\ -1_N & 0 \end{pmatrix}.$$

Denote by $C\left(\mathbb{T}^{2N}_{\theta}\right) \stackrel{\text{def}}{=} C\left(\mathbb{T}^{2N}_{\Theta}\right)$. Let $\{p_k \in \mathbb{N}\}_{k\in\mathbb{N}}$ be an infinite sequence of natural numbers such that $p_k > 1$ for any k, and let $m_j = \Pi^{j}_{k=1} p_k$. From the 5.4 it follows that there is a sequence of *-homomorphisms

$$\mathfrak{S}_\theta = \left\{C\left(\mathbb{T}^{2N}_{\theta}\right) \to C\left(\mathbb{T}^{2N}_{\theta/m_1^2}\right) \to C\left(\mathbb{T}^{2N}_{\theta/m_2^2}\right) \to ... \to C\left(\mathbb{T}^{2N}_{\theta/m_j^2}\right) \to ...\right\} \tag{5.60}$$

such that

(a) For any $j \in \mathbb{N}$ there are generators $u_{j-1,1}, ..., u_{j-1,2N} \in U\left(C\left(\mathbb{T}^{2N}_{\theta/m_{j-1}^2}\right)\right)$ and generators $u_{j,1}, ..., u_{j,2N} \in U\left(C\left(\mathbb{T}^{2N}_{\theta/m_j^2}\right)\right)$ such that the *-homomorphism $C\left(\mathbb{T}^{2N}_{\theta/m_{j-1}^2}\right) \to C\left(\mathbb{T}^{2N}_{\theta/m_j^2}\right)$ is given by

$$u_{j-1,k} \mapsto u^{p_j}_{j,k}; \ \ \forall k = 1, ..., 2N.$$

There are generators $u_1, ..., u_{2N} \in U\left(C\left(\mathbb{T}^{2N}_{\theta}\right)\right)$ such that *-homomorphism $C\left(\mathbb{T}^{2N}_{\theta}\right) \to C\left(\mathbb{T}^{2N}_{\theta/m_1^2}\right)$ is given by

$$u_j \mapsto u^{p_1}_{1,j}; \ \ \forall j = 1, ..., 2N,$$

(b) For any $j \in \mathbb{N}$ the triple $\left(C\left(\mathbb{T}^{2N}_{\theta/m_{j-1}^2}, C\left(\mathbb{T}^{2N}_{\theta/m_j^2}\right), \mathbb{Z}_{p_j}\right)\right)$ is a noncommutative finite-fold covering,

(c) There is the sequence of groups and epimorphisms

$$\mathbb{Z}^{2N}_{m_1} \leftarrow \mathbb{Z}^{2N}_{m_2} \leftarrow ...$$

which is equivalent to the sequence

$$G\left(C\left(\mathbb{T}^{2N}_{\theta/m_1^2}\right) \mid C\left(\mathbb{T}^{2N}_{\theta}\right)\right) \leftarrow G\left(C\left(\mathbb{T}^{2N}_{\theta/m_2^2}\right) \mid C\left(\mathbb{T}^{2N}_{\theta}\right)\right) \leftarrow ...$$
$$\leftarrow G\left(C\left(\mathbb{T}^{2N}_{\theta/m_j^2}\right) \mid C\left(\mathbb{T}^{2N}_{\theta}\right)\right) \leftarrow ... .$$

The sequence (5.60), is a specialization of (5.59), hence $\mathfrak{S}_\theta \in \mathfrak{FinAlg}$. Denote by $\widehat{C\left(\mathbb{T}^{2N}_{\theta}\right)} \stackrel{\text{def}}{=} \varinjlim C\left(\mathbb{T}^{2N}_{\theta/m_j^2}\right)$, $\widehat{G} \stackrel{\text{def}}{=} \varprojlim G\left(C\left(\mathbb{T}^{2N}_{\theta/m_j^2}\right) \mid C\left(\mathbb{T}^{2N}_{\theta}\right)\right)$. The group $\widehat{G}$ is Abelian because it is the inverse limit of Ablelian groups. Denote by $0_{\widehat{G}}$ (resp. "+") the neutral element of $\widehat{G}$ (resp. the product operation of $\widehat{G}$).

5.6.1 Inverse noncommutative limit

There are the equivariant representation

$$\widehat{\pi}^{\oplus}: \widehat{C\left(\mathbb{T}^{2N}_{\theta}\right)} \to \bigoplus_{g\in J} gL^2\left(\mathbb{R}^{2N}_{\theta}\right) \tag{5.61}$$

and an inclusion $\mathbb{Z}^{2N} \to \widehat{G}$ described in [12].

Theorem 5.26. *[12] Following conditions hold:*

(i) The representation $\widehat{\pi}^{\oplus}$ is good,

(ii)

$$\varprojlim_{\widehat{\pi}^{\oplus}} \downarrow \mathfrak{S}_{\theta} = C_0\left(\mathbb{R}^{2N}_{\theta}\right),$$

$$G\left(\varprojlim_{\widehat{\pi}^{\oplus}} \downarrow \mathfrak{S}_{\theta} \mid C\left(\mathbb{T}^{2N}_{\theta}\right)\right) = \mathbb{Z}^{2N}.$$

(iii) The triple $\left(C\left(\mathbb{T}^{2N}_{\theta}\right), C_0\left(\mathbb{R}^{2N}_{\theta}\right), \mathbb{Z}^{2N}\right)$ is an infinite noncommutative covering of $\mathfrak{S}_{\theta}$ with respect to $\widehat{\pi}^{\oplus}$.

5.6.2 Induced representation

Denote by $L^2\left(C_0\left(\mathbb{R}^{2N}_{\theta}\right)\right) \subset C_0\left(\mathbb{R}^{2N}_{\theta}\right)$ the space of square-summable elements (cf. Definition 3.19). Clearly $\mathcal{S}\left(\mathbb{R}^{2N}_{\theta}\right) \subset L^2\left(C_0\left(\mathbb{R}^{2N}_{\theta}\right)\right)$ and since $\mathcal{S}\left(\mathbb{R}^{2N}_{\theta}\right)$ is dense in $L^2\left(\mathbb{R}^{2N}_{\theta}\right)$ in the topology of the Hilbert space, $L^2\left(C_0\left(\mathbb{R}^{2N}_{\theta}\right)\right)$ is also dense in $L^2\left(\mathbb{R}^{2N}_{\theta}\right)$. Similarly to (3.7) we consider following pre-Hilbert space

$$L^2\left(C_0\left(\mathbb{R}^{2N}_{\theta}\right)\right) \otimes_{C\left(\mathbb{T}^{2N}_{\theta}\right)} L^2\left(C\left(\mathbb{T}^{2N}_{\theta}\right), \tau\right)$$

and denote by $\widetilde{\mathcal{H}}$ its Hilbert completion. From the dense inclusions $\mathcal{S}\left(\mathbb{R}^{2N}_{\theta}\right) \subset L^2\left(C_0\left(\mathbb{R}^{2N}_{\theta}\right)\right)$, $C^{\infty}\left(\mathbb{T}^{2N}_{\theta}\right) \subset L^2\left(C\left(\mathbb{T}^{2N}_{\theta}\right), \tau\right)$ it follows that the composition

$$\mathcal{S}\left(\mathbb{R}^{2N}_{\theta}\right) \otimes_{C^{\infty}\left(\mathbb{T}^{2N}_{\theta}\right)} C^{\infty}\left(\mathbb{T}^{2N}_{\theta}\right) \subset L^2\left(C_0\left(\mathbb{R}^{2N}_{\theta}\right)\right) \otimes_{C\left(\mathbb{T}^{2N}_{\theta}\right)} L^2\left(C\left(\mathbb{T}^{2N}_{\theta}\right)\right) \subset \widetilde{\mathcal{H}}$$

is the dense inclusion. Otherwise $\mathcal{S}\left(\mathbb{R}^{2N}_{\theta}\right) \otimes_{C^{\infty}\left(\mathbb{T}^{2N}_{\theta}\right)} C^{\infty}\left(\mathbb{T}^{2N}_{\theta}\right) \cong \mathcal{S}\left(\mathbb{R}^{2N}_{\theta}\right)$ it follows that there is the dense (with respect to the topology of the Hilbert space) inclusion

$$\mathcal{S}\left(\mathbb{R}^{2N}_{\theta}\right) \subset \widetilde{\mathcal{H}}.$$

$\widetilde{a}, \widetilde{b} \in \mathcal{S}\left(\mathbb{R}^{2N}_\theta\right)$ then from (3.7) it turns out it turns out

$$\left(\widetilde{a} \otimes 1_{C\left(\mathbb{T}^{2N}_\theta\right)}, \widetilde{b} \otimes \Psi_\theta\left(1_{C\left(\mathbb{T}^{2N}_\theta\right)}\right)\right)_{\widetilde{\mathcal{H}}} =$$
$$= \left(1_{C\left(\mathbb{T}^{2N}_\theta\right)}, \sum_{g \in \mathbb{Z}^{2N}} g\left(\widetilde{a}^* \widetilde{b}\right) \Psi_\theta\left(1_{C\left(\mathbb{T}^{2N}_\theta\right)}\right)\right)_{L^2\left(C\left(\mathbb{T}^{2N}_\theta\right), \tau\right)} =$$
$$= \int_{\mathbb{T}^{2N}} \left(\sum_{g \in \mathbb{Z}^{2N}} g\left(\widetilde{a}^*_{\text{comm}} \widetilde{b}_{\text{comm}}\right)\right)(x)\ dx = \int_{\mathbb{R}^{2N}} \widetilde{a}^*_{\text{comm}} \widetilde{b}_{\text{comm}}\left(\widetilde{x}\right)\ d\widetilde{x}$$

where $\widetilde{a}_{\text{comm}} \in \mathcal{S}\left(\mathbb{R}^{2N}\right)$ (resp. $\widetilde{b}_{\text{comm}} \in \mathcal{S}\left(\mathbb{R}^{2N}\right)$) is a commutative function which corresponds to $\widetilde{a}$ (resp. $\widetilde{b}$). Above equation coincides with (5.37). Taking into account that $\mathcal{S}\left(\mathbb{R}^{2N}_\theta\right)$ is dense in $L^2\left(\mathbb{R}^{2N}_\theta\right)$ one has an isomorphism

$$\widetilde{\mathcal{H}} \approx L^2\left(\mathbb{R}^{2N}_{\widetilde{\theta}}\right)$$

of Hilbert spaces. Thus if $\rho : C\left(\mathbb{T}^{2N}_\theta\right) \to L^2\left(C\left(\mathbb{T}^{2N}_\theta\right), \tau\right)$ then both

$$\widehat{\rho} : \widehat{C\left(\mathbb{T}^{2N}_\theta\right)} \to B\left(L^2\left(\mathbb{R}^{2N}_{\widetilde{\theta}}\right)\right),$$
$$\widetilde{\rho} : C_0\left(\mathbb{R}^{2N}_\theta\right) \to B\left(L^2\left(\mathbb{R}^{2N}_{\widetilde{\theta}}\right)\right)$$

are induced by $\left(\rho, \mathfrak{S}_\theta, \widehat{\pi}^\oplus\right)$.

5.6.3 The sequence of spectral triples

Let us consider following objects

- A spectral triple of a noncommutative torus $\left(C^\infty(\mathbb{T}^{2N}_\theta), \mathcal{H} = L^2\left(C\left(\mathbb{T}^{2N}_\theta\right), \tau\right) \otimes \mathbb{C}^{2^N}, D\right)$,
- A good algebraical finite covering sequence given by

$$\mathfrak{S}_\theta = \left\{C\left(\mathbb{T}^{2N}_\theta\right) \to C\left(\mathbb{T}^{2N}_{\theta/m_1^2}\right) \to C\left(\mathbb{T}^{2N}_{\theta/m_2^2}\right) \to ... \to C\left(\mathbb{T}^{2N}_{\theta/m_j^2}\right) \to ...\right\} \in \mathfrak{FinAlg}.$$

  given by (5.60).

Otherwise from the Theorem 5.5 it follows that

$$\mathfrak{S}_{\left(C^\infty(\mathbb{T}_\theta), L^2\left(C\left(\mathbb{T}^{2N}_\theta\right), \tau\right) \otimes \mathbb{C}^{2^N}, D\right)} = \{\left(C^\infty\left(\mathbb{T}_\theta\right), L^2\left(C\left(\mathbb{T}^{2N}_\theta\right), \tau\right) \otimes \mathbb{C}^{2^N}, D\right), \dots,$$
$$\left(C^\infty\left(\mathbb{T}^{2N}_{\theta/m_j^2}\right), L^2\left(C\left(\mathbb{T}^{2N}_{\theta/m_j^2}\right), \tau_j\right) \otimes \mathbb{C}^{2^N}, D_j\right), \dots\} \in \mathfrak{CohTriple} \quad (5.62)$$

is a coherent sequence of spectral triples. We would like to proof that

$$\mathfrak{S}_{\left(C^\infty(\mathbb{T}_\theta), L^2\left(C\left(\mathbb{T}^{2N}_\theta\right), \tau\right) \otimes \mathbb{C}^{2^N}, D\right)}$$

is regular and to find a $\left(C\left(\mathbb{T}^{2N}_\theta\right), C\left(\mathbb{R}^{2N}_\theta\right), \mathbb{Z}^{2N}\right)$-lift of $\left(C^\infty\left(\mathbb{T}_\theta\right), L^2\left(C\left(\mathbb{T}^{2N}_\theta\right), \tau\right) \otimes \mathbb{C}^{2^N}\right)$. If $\rho : C\left(\mathbb{T}_\theta\right) \to B\left(L^2\left(C\left(\mathbb{T}^{2N}_\theta\right), \tau\right)\right) \otimes \mathbb{C}^{2^N}$ is the natural representation then from the 5.6.2 it turns out that

$$\widetilde{\rho} : C\left(\mathbb{R}^{2N}_\theta\right) \to B\left(L^2\left(\mathbb{R}^{2N}_\theta\right) \otimes \mathbb{C}^{2^N}\right)$$

is induced by $(\rho, \mathfrak{S}_\theta, \widehat{\pi}^\oplus)$. Let us consider a topological covering $\varphi : \mathbb{R}^{2N} \to \mathbb{T}^{2N}$ and a commutative spectral triple $\left(C^\infty\left(\mathbb{T}^{2N}\right), L^2\left(\mathbb{T}^{2N}, S\right), \not{D}\right)$ given by (5.20). Denote by $\widetilde{S} = \varphi^* S$, $\widetilde{\not{D}} = \varphi^* \not{D}$ inverse images of the Spin-bundle S and the Dirac operator $\not{D}$ (cf. 1.2.2, 1.13). Otherwise there is a natural isomorphism of Hilbert spaces

$$\widetilde{\varphi} : L^2\left(\mathbb{R}^{2N}_\theta\right) \otimes \mathbb{C}^{2^N} \xrightarrow{\approx} L^2\left(\mathbb{R}^{2N}\right) \otimes \mathbb{C}^{2^N}.$$

Denote by

$$\widetilde{D} = \widetilde{\varphi}^{-1} \circ \widetilde{\not{D}} \circ \widetilde{\varphi}.$$

5.6.4 Smooth elements

Following lemmas will be used for the construction of the smooth algebra.

Lemma 5.27. *[12] Following conditions hold:*

(i) Let $\left\{a_n \in C_b\left(\mathbb{R}^{2N}_\theta\right)\right\}_{n \in \mathbb{N}}$ be a sequence such that

- *$\{a_n\}$ is weakly-* convergent (cf. Definition 5.1),*
- *If $a = \lim_{n\to\infty} a_n$ in the sense of weak-* convergence then $a \in C_b\left(\mathbb{R}^{2N}_\theta\right)$.*

Then the sequence $\{a_n\}$ is convergent in sense of weak topology $\{a_n\}$ (cf. Definition 1.37) and a is limit of $\{a_n\}$ with respect to the weak topology. Moreover if $\{a_n\}$ is increasing or decreasing sequence of self-adjoint elements then $\{a_n\}$ is convergent in sense of strong topology (cf. Definition 1.36) and a is limit of $\{a_n\}$ with respect to the strong topology.

(ii) If $\{a_n\}$ is strongly and/or weakly convergent (cf. Definitions 1.36, 1.37) and $a = \lim_{n\to\infty} a_n$ is strong and/ or weak limit then $\{a_n\}$ is weakly- convergent and a is the limit of $\{a_n\}$ in the sense of weakly-* convergence.*

Lemma 5.28. *[12] Let $\overline{G}_j = \ker\left(\mathbb{Z}^{2N} \to \mathbb{Z}^{2N}_{m_j}\right)$. Let $\widetilde{a} \in \mathcal{S}\left(\mathbb{R}^{2N}_\theta\right)$ and let*

$$a_j = \sum_{g \in \overline{G}_j} g\widetilde{a} \qquad (5.63)$$

where the sum the series means weakly- convergence. Following conditions hold:*

(i) $a_j \in C^\infty\left(\mathbb{R}^{2N}\right)$,

(ii) The series (5.63) is convergent with respect to the strong topology (cf. Definition 1.36),

(iii) There is a following strong limit

$$\widetilde{a} = \lim_{j\to\infty} a_j. \tag{5.64}$$

Lemma 5.29. *The system of seminorms $\|\cdot\|_s$ given by* (3.11) *is equivalent to the system of seminorms $\|\cdot\|_{(t_1,\dots,t_{2N})}$ given by*

$$\|\widetilde{a}\|_{(t_1,\dots,t_{2N})} \overset{\text{def}}{=} \left\| \frac{\partial^{t_1+\dots+t_{2N}}}{\partial x_1^{t_1}\dots\partial x_{2N}^{t_{2N}}}\widetilde{a} \right\|_{\text{op}}$$

where $t_1, \dots t_{2N} \in \mathbb{N}^0$, and $\frac{\partial^{t_1+\dots+t_{2N}}}{\partial x_1^{t_1}\dots\partial x_{2N}^{t_{2N}}}$ means partial derivation of $\widetilde{a}$ regarded as element in $\mathcal{S}'\left(\mathbb{R}^{2N}\right)$ and $\|\cdot\|_{\text{op}}$ is the operator norm given by (5.39).

Proof. Operators $1_{C_b\left(\mathbb{R}^{2N}_\theta\right)} \otimes \pi^s_j\left(a_j\right) \in B\left(\widetilde{\mathcal{H}}^{2^s}\right)$ from the condition (b) of the Definition 3.31 can be regarded as matrices in $\mathbb{M}_{2^s 2^N}\left(B\left(L^2\left(\mathbb{R}^{2N}\right)\right)\right)$, so we will write

$$1_{C_b\left(\mathbb{R}^{2N}_\theta\right)} \otimes \pi^s_j\left(a_j\right) = \left(m^j_{\alpha\beta}\right)_{\alpha,\beta=1,\dots 2^s 2^N} \in \mathbb{M}_{2^s 2^N}\left(B\left(L^2\left(\mathbb{R}^{2N}\right)\right)\right)$$

From (5.29) it follows that

$$\left[D_j, a_j\right] = \sum_{\mu=1}^{2N} \frac{\partial a_j}{\partial x_\mu} u^*_\mu \left[D, u_\mu\right] = \sum_{\mu=1}^{2N} \gamma^\mu \frac{\partial a_j}{\partial x_\mu} u^*_\mu \left[\delta_\mu, u_\mu\right] \tag{5.65}$$

where $u_1, \dots, u_n$ are unitary generators of $C^\infty\left(\mathbb{T}^n_\theta\right)$ If $s = 1$ then from (5.18) (5.29) it follows that for any α, β element $m_{\alpha\beta}$ is given by

$$m^j_{\alpha\beta} = a_j$$

or there is $\mu \in \{1, \dots, 2N\}$ such that

$$m^j_{\alpha\beta} = \frac{\partial a_j}{\partial x_\mu} u^*_\mu \left[\delta_\mu, u_\mu\right] \tag{5.66}$$

and taking into account (5.26) one has

$$m^j_{\alpha\beta} = \frac{\partial a_j}{\partial x_\mu}. \tag{5.67}$$

From $\widetilde{a} \in \mathcal{S}\left(\mathbb{R}^{2N}_\theta\right)$ and the Lemma 5.28 for any $\mu = 1, \dots 2N$ the sequence

$$\left\{ \frac{\partial a_j}{\partial x_\mu} \in C^\infty\left(\mathbb{T}^{2N}_{\theta/m_j^2}\right) \right\}_{j\in\mathbb{N}}$$

is strongly convergent and following condition holds

$$\lim_{j\to\infty} \frac{\partial a_j}{\partial x_\mu} = \frac{\partial \widetilde{a}}{\partial x_\mu}.$$

Hence if $m^j_{\alpha\beta}$ is given by (5.66) then there is a following strong limit

$$\lim_{j\to\infty} m^j_{\alpha\beta} = \widetilde{m}_{\alpha\beta} = \frac{\partial \widetilde{a}}{\partial x_\mu} u^*_\mu \left[\delta_\mu, u_\mu\right] = \frac{\partial \widetilde{a}}{\partial x_\mu} \tag{5.68}$$

where $\widetilde{m}_{\alpha\beta}$ is element of matrix which represent $\pi^1\left(\widetilde{a}\right)$. From (5.65) and the Lemma 5.28 one has a strong limit

$$\lim_{j\to\infty} 1_{C_b\left(\mathbb{R}^{2N}\right)} \otimes \left[D_j, a_j\right] = \sum_{\mu=1}^{2N} \frac{\partial \widetilde{a}}{\partial x_\mu} u^*_\mu \left[D, u_\mu\right] = \sum_{\mu=1}^{2N} \gamma^\mu \frac{\partial \widetilde{a}}{\partial x_\mu}. \tag{5.69}$$

It follows that from (1.4) and (5.66) if $s = 2$ then the matrix which corresponds to $\pi^2_j\left(a_j\right)$ for any $\mu = 1, \dots, 2N$ contains a submatrix

$$\left[D_j, \frac{\partial a_j}{\partial x_\mu}\right].$$

For any $\nu = 1, \dots, 2N$ the above submatrix contains an element given by

$$m^j_{\alpha\beta} = \frac{\partial^2 a_j}{\partial x_\nu \partial x_\mu} u^*_\nu \left[\delta_\nu, u_\nu\right] = \frac{\partial^2 a_j}{\partial x_\nu \partial x_\mu}. \tag{5.70}$$

From the and the Lemma 5.28 one has a strong limit

$$\lim_{j\to\infty} \frac{\partial^2 a_j}{\partial x_\nu \partial x_\mu} = \frac{\partial^2 \widetilde{a}}{\partial x_\nu \partial x_\mu},$$

so the matrix $\left\{m_{\alpha\beta}\right\}$ contains an element $\frac{\partial^2 \widetilde{a}}{\partial x_\nu \partial x_\mu}$. Similarly for any multiindex $\left(t_1, ..., t_{2N}\right) \in \left(\mathbb{N}^0\right)^{2M}$ there is $s \in \mathbb{N}$ such that $1_{C_b\left(\mathbb{R}^{2N}_\theta\right)} \otimes \pi^s_j\left(a_j\right) \in B\left(\widetilde{\mathcal{H}}^{2^s}\right)$ is represented by a matrix

$$\left(m^j_{\alpha\beta}\right)_{\alpha,\beta=1,\dots 2^s 2^N} \in \mathbb{M}_{2^s 2^N}\left(B\left(L^2\left(\mathbb{R}^{2N}\right)\right)\right)$$

such that there are α, β such that

$$m^j_{\alpha\beta} = \frac{\partial^{t_1+\dots+t_{2N}} a_j}{\partial x_1^{t_1} \dots \partial x_{2N}^{t_{2N}}}. \tag{5.71}$$

From the and the Lemma 5.28 it follows that for any $s \in \mathbb{N}$ there are strong limits

$$\widetilde{m}_{\alpha\beta} = \lim_{j\to\infty} m^j_{\alpha\beta} = \frac{\partial^{t_1+\dots+t_{2N}} \widetilde{a}}{\partial x_1^{t_1} \dots \partial x_{2N}^{t_{2N}}}, \tag{5.72}$$

so one has the strong limit $\pi^s\left(\widetilde{a}\right) = \lim_{j\to\infty} \pi^s\left(a_j\right)$ for any $s \in \mathbb{N}$. The operator $1_{C_b\left(\mathbb{R}^{2N}_\theta\right)} \otimes \pi^s_j\left(a_j\right) \in B\left(\widetilde{\mathcal{H}}^{2^s}\right)$ is represented by a matrix $\left(m^j_{\alpha\beta}\right)_{\alpha,\beta=1,\dots 2^s 2^N} \in \mathbb{M}_{2^s 2^N}\left(B\left(L^2\left(\mathbb{R}^{2N}\right)\right)\right)$ it follows that the norm $\|\pi^s(\widetilde{a})\|$ of $\pi^s(\widetilde{a})$ is equivalent to the system of operator norms of its matrix elements given by

$$\|\widetilde{m}_{\alpha\beta}\| = \left\| \frac{\partial^{t_1+\dots+t_{2N}}\widetilde{a}}{\partial x_1^{t_1}\dots\partial x_{2N}^{t_{2N}}} \right\|_{\mathrm{op}}. \tag{5.73}$$

□

Lemma 5.30. *Any $\widetilde{a} \in \mathcal{S}\left(\mathbb{R}^{2N}_\theta\right)$ satisfies to the conditions (b), (c) of the Definition 3.31.*

Proof. (b) Follows from the Lemma 5.29.
(c) From (5.69) it turns out that

$$\lim_{j\to\infty} 1_{C_b\left(\mathbb{R}^{2N}_\theta\right)} \otimes \left[D_j, a_j\right] = \sum_{\mu=1}^{2N} \frac{\partial \widetilde{a}}{\partial x_\mu} u^*_\mu \left[D, u_\mu\right] \tag{5.74}$$

If $L^2\left(C_0\left(\mathbb{R}^{2N}_\theta\right)\right)$ is a space of square-summable elements (cf. Definition 3.19) then $\mathcal{S}\left(\mathbb{R}^{2N}_\theta\right) \subset L^2\left(C_0\left(\mathbb{R}^{2N}_\theta\right)\right)$. Taking into account $\frac{\partial \widetilde{a}}{\partial x_\mu} \in \mathcal{S}\left(\mathbb{R}^{2N}_\theta\right)$, $u^*_\mu\left[D, u_\mu\right] \in \Omega_D$ and (5.74) one has

$$\lim_{j\to\infty} 1_{C_b(\mathbb{R}^{2n})} \otimes \left[D_j, a_j\right] \in L^2\left(C_0\left(\mathbb{R}^{2N}_\theta\right)\right) \otimes_{C\left(\mathbb{T}^{2N}_\theta\right)} \Omega^1_D.$$

□

Corollary 5.31. *Let $f_{mn} \in \mathcal{S}\left(\mathbb{R}^2_\theta\right)$ be given by the Proposition 5.16. If $\widetilde{a} \in \mathcal{S}\left(\mathbb{R}^{2N}_\theta\right)$ is such that*

$$\widetilde{a} = f_{m_1 n_1} \otimes \dots \otimes f_{m_N n_N}; \quad \text{(cf. (5.45))} \tag{5.75}$$

then $\widetilde{a}$ is a $\mathfrak{S}_{\left(C^\infty(\mathbb{T}_\theta), L^2\left(C\left(\mathbb{T}^{2N}_\theta\right), \tau\right) \otimes \mathbb{C}^{2^N}, D\right)}$-smooth element with respect to $\widehat{\pi}^\oplus$.

Proof. From the Proposition 5.16 it turns out that $f_{m_1 n_1}, \dots, f_{m_N n_N}$ are rank-one operators, hence $\widetilde{a} = f_{m_1 n_1} \otimes \dots \otimes f_{m_N n_N}$ is also a rank-one operator. So $\widetilde{a}$ lies in the Pederesen ideal of $C_0\left(\mathbb{R}^{2N}_\theta\right)$, i.e. $\widetilde{a}$ satisfies to the condition (d) of the Definition 3.31). The conditions (a) follows from the Lemma 5.28 conditions (b), (c) follow from the Lemma (5.30)

□

5.32. If $\widetilde{a}$ is a $\mathfrak{S}_{\left(C^\infty(\mathbb{T}_\theta), L^2\left(C\left(\mathbb{T}^{2N}_\theta\right), \tau\right) \otimes \mathbb{C}^{2^N}, D\right)}$-smooth element with respect to $\widehat{\pi}^\oplus$ then from (5.74) it follows that if $\widetilde{a}_D$ is given by (3.12) then

$$\widetilde{a}_D = \lim_{j\to\infty} 1_{C_b(\mathbb{R}^{2n})} \otimes \left[D_j, a_j\right] = \sum_{\mu=1}^{2N} \frac{\partial \widetilde{a}}{\partial x_\mu} u^*_\mu \left[D, u_\mu\right] \tag{5.76}$$

For any $\xi = (\xi_1, \dots, \xi_{2^N}) \in C^\infty\left(\mathbb{T}^{2N}_\Theta\right) \otimes \mathbb{C}^{2^N} \subset L^2\left(C^\infty\left(\mathbb{T}^{2N}_\Theta\right), \tau\right) \otimes \mathbb{C}^{2^N}$ the operator $\widetilde{D}$ given by (3.13) satisfies to the following condition

$$\widetilde{D}\left(\widetilde{a} \otimes \xi\right) = \sum_{\mu=1}^{2N} \frac{\partial \widetilde{a}}{\partial x_\mu} u^*_\mu \left[D, u_\mu\right] \xi + \widetilde{a} D \xi = \sum_{\mu=1}^{2N} \frac{\partial \widetilde{a}}{\partial x_\mu} \otimes \gamma^\mu \xi + \widetilde{a} \otimes \sum_{\mu=1}^{2N} \gamma^\mu \frac{\partial \xi}{\partial x_\mu},$$

and taking into account (5.58) one has.

$$\widetilde{D} = \sum_{\mu=1}^{2N} \gamma^\mu \frac{\partial}{\partial x_\mu}. \tag{5.77}$$

The given by (3.11) seminorms $\|\cdot\|_s$ satisfy to following equation

$$\|\widetilde{a}\|_s = \|\pi^s(\widetilde{a})\|_{\mathrm{op}} \tag{5.78}$$

where

$$\pi^s(\widetilde{a}) = \begin{pmatrix} \pi^{s-1}(\widetilde{a}) & 0 \\ \left[\sum_{\mu=1}^{2N} \gamma^\mu \frac{\partial}{\partial x_\mu}, \pi^{s-1}(\widetilde{a})\right] & \pi^{s-1}(\widetilde{a}) \end{pmatrix}.$$

Lemma 5.33. *Let $\widetilde{a}$ be a $\mathfrak{S}_{\left(C^\infty(\mathbb{T}_\theta), L^2\left(C\left(\mathbb{T}^{2N}_\theta\right), \tau\right) \otimes \mathbb{C}^{2N}, D\right)}$-smooth element with respect to $\widehat{\pi}^\oplus$ given by* (5.61) *(cf. Definition 3.31) then $\widetilde{a} \in \mathcal{S}\left(\mathbb{R}^{2N}_\theta\right)$.*

Proof. Let $\widetilde{a}$ is a $\mathfrak{S}_{\left(C^\infty(\mathbb{T}_\theta), L^2\left(C\left(\mathbb{T}^{2N}_\theta\right), \tau\right) \otimes \mathbb{C}^{2N}, D\right)}$-smooth element with respect to $\widehat{\pi}^\oplus$ given by (5.61) (cf. 3.31). From the condition (d) of the Definition 3.31) and the Theorem 5.26 it follows that $\widetilde{a} \in K\left(C_0\left(\mathbb{R}^{2N}_\theta\right)\right)$. Otherwise from $C_0\left(\mathbb{R}^{2N}_\theta\right) \approx \mathcal{K}$ (cf. (5.45)) and taking into account that any $b \in K(\mathcal{K})$ is a finite-rank operator, one concludes that $\widetilde{a}$ is a finite-rank operator. From this fact and (5.45) it turns out that

$$\widetilde{a} = \sum_{j=1}^{M} \widetilde{a}^j, \text{ where } \widetilde{a}^j = \widetilde{a}^j_1 \otimes \dots \otimes \widetilde{a}^j_N \in \underbrace{C\left(\mathbb{R}^2_\theta\right) \otimes \dots \otimes C\left(\mathbb{R}^2_\theta\right)}_{N-\text{times}}, \tag{5.79}$$

where $\widetilde{a}^j_1, \dots \widetilde{a}^j_N \in C\left(\mathbb{R}^2_\theta\right)$ are finite-rank operators. Let us select the representation (5.79) such that M is minimal. If $a \in C_0\left(\mathbb{R}^2_\theta\right)$ is a finite-rank operator then it can be represented by the following matrix

$$a = u \begin{pmatrix} \lambda_1 & 0 & \dots & 0 & 0 & \dots \\ 0 & \lambda_2 & \dots & 0 & 0 & \dots \\ \vdots & \vdots & \ddots & \vdots & \vdots & \dots \\ 0 & 0 & \dots & \lambda_r & 0 & \dots \\ 0 & 0 & \dots & 0 & 0 & \dots \\ \vdots & \vdots & \dots & \vdots & \vdots & \ddots \end{pmatrix} v \tag{5.80}$$

where u, v are finite-rank partial isometries. Above operator can be represented by following way

$$a = \sum_{k=1}^{r} \alpha_k \beta_k, \text{ where } \alpha_k, \beta_k \text{ are given by}$$

$$\begin{aligned}\alpha_k &= u \begin{pmatrix} 0 & \dots & 0 & 0 & \dots \\ \vdots & \ddots & \vdots & \vdots & \dots \\ 0 & \dots & |\lambda_k| & 0 & \dots \\ 0 & \dots & 0 & 0 & \dots \\ \vdots & \vdots & \dots & \vdots & \ddots \end{pmatrix}, \\ \beta_k &= \begin{pmatrix} 0 & \dots & 0 & 0 & \dots \\ \vdots & \ddots & \vdots & \vdots & \dots \\ 0 & \dots & \frac{\lambda_k}{|\lambda_k|} & 0 & \dots \\ 0 & \dots & 0 & 0 & \dots \\ \vdots & \vdots & \dots & \vdots & \ddots \end{pmatrix} v.\end{aligned} \tag{5.81}$$

Above equation is equivalent to

$$\begin{aligned}\alpha_k &= \sum_{m=0}^{\infty} \alpha_{mk} f_{mk}, \\ \beta_k &= \sum_{m=0}^{\infty} \beta_{km} f_{km}\end{aligned} \tag{5.82}$$

where $\left\{f_{mk} \in \mathcal{S}\left(\mathbb{R}^2_\theta\right)\right\}_{m,k\in\mathbb{N}^0}$ are given by the Proposition 5.16. From the above equation it follows that $\alpha_k\beta_k$ is a bounded operator if and only if

$$\sum_{m=0}^{\infty} |\alpha_{mk}|^2 < \infty,$$
$$\sum_{m=0}^{\infty} |\beta_{km}|^2 < \infty.$$

From the Remark 5.22 it turns out $\alpha_k\beta_k \in C_0\left(\mathbb{R}^2\right)$. From this fact and taking to account equation (5.81) one concludes that any term of the finite sum (5.79) lies in $C_0\left(\mathbb{R}^2\right)$. It follows that $\widetilde{a} \in C_0\left(\mathbb{R}^{2N}\right)$. Denote by

$$a_j = \sum_{g\in\ker\left(\mathbb{Z}^{2N}\to\mathbb{Z}^{2N}_{m_j}\right)} \widetilde{a}.$$

From the condition (a) of the Definition 3.31 it turns out that $a_j \in C^\infty\left(\mathbb{T}^{2N}_{\theta/m_j}\right)$. It follows that a_j corresponds to a smooth function $a_j \in C^\infty\left(\mathbb{T}^{2N}_{m_j}\right)$. Otherwise a_j can be regarded as a smooth periodic function on $\mathbb{R}^{2N}$ and a_j is a distribution, i.e. $a_j \in \mathcal{S}'\left(\mathbb{R}^{2N}\right)$. From the proof of the Lemma 5.29 one can write

$$1_{C_b\left(\mathbb{R}^{2N}_\theta\right)} \otimes \pi^s_j\left(a_j\right) = \left(m^j_{\alpha\beta}\right)_{\alpha,\beta=1,\dots 2^s 2^N} \in \mathbb{M}_{2^s 2^N}\left(B\left(L^2\left(\mathbb{R}^{2N}\right)\right)\right).$$

From (5.72) it follows that for any multiindex $(t_1, \dots t_{2N}) \in \left(\mathbb{N}^0\right)^{2N}$ there is $s \in \mathbb{N}$ such that $1_{C_b\left(\mathbb{R}^{2N}_\theta\right)} \otimes \pi^s_j\left(a_j\right) \in B\left(\widetilde{\mathcal{H}}^{2^s}\right)$ is represented by a matrix

$$\left(m^j_{\alpha\beta}\right)_{\alpha,\beta=1,\dots 2^s 2^N} \in \mathbb{M}_{2^s 2^N}\left(B\left(L^2\left(\mathbb{R}^{2N}\right)\right)\right)$$

such that there are α, β which satisfy to the following equation

$$m^j_{\alpha\beta} = \frac{\partial^{t_1+\dots+t_{2N}} a_j}{\partial x_1^{t_1} \dots \partial x_{2N}^{t_{2N}}}. \tag{5.83}$$

From the condition (b) of the Definition 3.31 following conditions hold:

- For any multiindex $(t_1, \dots, t_{2N}) \in \left(\mathbb{N}^0\right)^{2N}$ there is the following limit

  $$\widetilde{m}_{\alpha\beta} = \lim_{j\to\infty} m^j_{\alpha\beta}$$

  in the strong topology (cf. Definition 1.36).
- The limit corresponds to a bounded operator with respect to the operator norm (5.39).

From (ii) of the Lemma 5.27 it follows that the strong topology limit is $\widetilde{m}_{\alpha\beta} = \lim_{j\to\infty} m^j_{\alpha\beta}$ is the limit in sense of the weak-* convergence, so one has

$$\widetilde{m}_{\alpha\beta} = \lim_{j\to\infty} m^j_{\alpha\beta} = \frac{\partial^{t_1+\dots+t_{2N}} \widetilde{a}}{\partial x_1^{t_1} \dots \partial x_{2N}^{t_{2N}}}. \tag{5.84}$$

and right part of (5.84) corresponds to a bounded operator, i.e. $\frac{\partial^{t_1+\dots+t_{2N}} \widetilde{a}}{\partial x_1^{t_1} \dots \partial x_{2N}^{t_{2N}}} \in B\left(L^2\left(\mathbb{R}^{2N}\right)\right)$. If one considers a factorization (5.79)

$$\widetilde{a} = \sum_{j=1}^{M} \widetilde{a}^j, \text{ where } \widetilde{a}^j = \widetilde{a}^j_1 \otimes \dots \otimes \widetilde{a}^j_N \in \underbrace{C\left(\mathbb{R}^2_\theta\right) \otimes \dots \otimes C\left(\mathbb{R}^2_\theta\right)}_{N-\text{times}}$$

such that M is minimal then all partial tensor products

$$P_j = \widetilde{a}^j_2 \otimes \dots \otimes \widetilde{a}^j_N \in \underbrace{C\left(\mathbb{R}^2_\theta\right) \otimes \dots \otimes C\left(\mathbb{R}^2_\theta\right)}_{(N-1)-\text{times}}; \;\; j = 1, \dots, M$$

are linearly independent. Similarly to 5.24 for j^{th} term of the tensor product

$$\underbrace{C\left(\mathbb{R}^2_\theta\right)\otimes\cdots\otimes C\left(\mathbb{R}^2_\theta\right)}_{N-\text{times}}$$

we denote by p_j, q_j coordinates which satisfy to (5.51) - (5.52). One has

$$\frac{\partial}{\partial p_1}\widetilde{a} = \sum_{j=1}^{M}\frac{\partial}{\partial p_1}\widetilde{a}^j_1\otimes\widetilde{a}^j_2\otimes\cdots\otimes\widetilde{a}^j_N.$$

Elements P_j are linearly independent, it follows that if any term in the above sum is unbounded with respect to the norm (5.39) then all sum is unbounded. So $\frac{\partial}{\partial p_1}\widetilde{a}^j_1$ is bounded for any $j = 1, \dots M$. Otherwise $\widetilde{a}^1_1$ is a finite-rank operator it follows that $\widetilde{a}^1_1$ can be represented by (5.81), i.e.

$$\widetilde{a}^1_1 = \sum_{k=1}^{r}\alpha_k\times\beta_k \tag{5.85}$$

Otherwise taking into account (5.53) one has

$$\frac{\partial}{\partial p_1}\widetilde{a}^j_1 = \sum_{k=1}^{r}-iq_1\times\alpha_k\times\beta_k + \sum_{k=1}^{r}\alpha_k\times\beta_k\times iq_1. \tag{5.86}$$

From (5.81) it turns out that all terms in (5.86) are linearly independent, so if one or more terms are unbounded (with respect to the norm (5.39)) then the whole sum is unbounded. Otherwise $q_1\times\alpha_k\times\beta_k$ is unbounded if and only if $q_1\times\alpha_k$ is unbounded. Similarly $\alpha_k\times\beta_k\times q_1$ is unbounded if and only if $\beta_k\times q_1$ is unbounded. From this fact it turns out that all operators

$$q_1\times\alpha_k,\quad \beta_k\times q_1$$

are bounded. Similarly one can prove that following operators

$$p_1\times\alpha_k,\quad \beta_k\times p_1$$

are bounded. Clearly if

$$a_1 = \frac{q_1+ip_1}{\sqrt{2}};\ \ \overline{a}_1 = \frac{q_1-ip_1}{\sqrt{2}}$$

then operators

$$a_1\times\alpha_k,\quad \beta_k\times a_1,\ \overline{a}_1\times\alpha_k,\quad \beta_k\times\overline{a}_1,$$

are bounded. Similarly to (5.54) we define $H_1 = \overline{a}_1 \times a_1 - 1$. For any $m, n \in \mathbb{N}$ a distribution $\frac{\partial^m}{\partial^m p_1} \frac{\partial^n}{\partial^n q_1} \widetilde{a}_1^1$ is a bounded (with respect to (5.39)) it follows that for any $l \in \mathbb{N}$ following distributions

$$\underbrace{H_1 \times \cdots \times H_1}_{l-\text{times}} \times \alpha_k, \quad \beta_k \times \underbrace{H_1 \times \cdots \times H_1}_{l-\text{times}}$$

are bounded operators. From (5.56) and (5.82) it follows that

$$\alpha_k = \sum_{m=0}^{\infty} \alpha_{mk} f_{mk},$$
$$\beta_k = \sum_{m=0}^{\infty} \beta_{km} f_{km},$$
$$\underbrace{H_1 \times \cdots \times H_1}_{l-\text{times}} \times \alpha_k = \sum_{m=0}^{\infty} (2m+1)^l \alpha_{mk} f_{mk},$$
$$\beta_k \times \underbrace{H_1 \times \cdots \times H_1}_{l-\text{times}} = \sum_{m=0}^{\infty} (2m+1)^l \beta_{km} f_{km},$$

hence operators $\underbrace{H_1 \times \cdots \times H_1}_{l-\text{times}} \times \alpha_k$ and $\beta_k \times \underbrace{H_1 \times \cdots \times H_1}_{l-\text{times}}$ are bounded if following conditions hold:

$$\sum_{m=0}^{\infty} (2m+1)^{2l} |\alpha_{mk}|^2 < \infty,$$
$$\sum_{m=0}^{\infty} (2m+1)^{2l} |\beta_{km}|^2 < \infty$$

Form the Definition 5.21 it follows that for any $s \in \mathbb{N}$ following conditions hold:

$$\alpha_k \in \mathcal{G}_{2l,s}, \quad \beta_k \in \mathcal{G}_{s,2l}$$

Since we can select arbitrary l and taking into account (5.50) one has

$$\alpha_k \times \beta_k \in \mathcal{S}\left(\mathbb{R}^2\right).$$

From (5.85) it turns out that

$$\widetilde{a}_1^1 \in \mathcal{S}\left(\mathbb{R}^2\right)$$

If we consider representation (5.79)

$$\widetilde{a} = \sum_{j=1}^{M} \widetilde{a}^j, \text{ where } \widetilde{a}^j = \widetilde{a}_1^j \otimes \cdots \otimes \widetilde{a}_N^j \in \underbrace{C\left(\mathbb{R}_\theta^2\right) \otimes \cdots \otimes C\left(\mathbb{R}_\theta^2\right)}_{N-\text{times}}$$

then similarly to the above construction one can prove that

$$\widetilde{a}^j_k \in \mathcal{S}\left(\mathbb{R}^2\right);\ j = 1, \dots, M,\ \ k = 1, \dots N.$$

From (5.79) it follows that

$$\widetilde{a} \in \underbrace{\mathcal{S}\left(\mathbb{R}^2_\theta\right) \otimes \dots \otimes \mathcal{S}\left(\mathbb{R}^2_\theta\right)}_{N-\text{times}} \subset \mathcal{S}\left(\mathbb{R}^{2N}\right).$$

where one means the algebraic tensor product. □

Lemma 5.34. *Denote by* $C^\infty_0\left(\mathbb{R}^{2N}_\theta\right)$ *the smooth algebra of* $\mathfrak{S}_{\left(C^\infty(\mathbb{T}_\theta), L^2\left(C\left(\mathbb{T}^{2N}_\theta\right), \tau\right)\otimes \mathbb{C}^{2^N}, D\right)}$ *with respect to* $\widehat{\pi}^\oplus$*. Following condition holds*

$$\mathcal{S}\left(\mathbb{R}^{2N}\right) \subset C_0\left(\mathbb{R}^{2N}_\theta\right).$$

Proof. Let $I_0 = \left(\mathbb{N}^0\right)^2$ and let $I = I_0^N$. For any $\nu = \left(\left(m^\nu_1, n^\nu_1\right), \dots, \left(m^\nu_N, n^\nu_N\right)\right) \in I$ we denote

$$f_\nu \overset{\text{def}}{=} f_{m^\nu_1 n^\nu_1} \otimes \dots \otimes f_{m^\nu_N n_N} \in \underbrace{\mathcal{S}\left(\mathbb{R}^2\right) \otimes \dots \otimes \mathcal{S}\left(\mathbb{R}^2\right)}_{N-\text{times}} \subset \mathcal{S}\left(\mathbb{R}^{2N}\right)$$

where we mean the algebraic tensor product. Indeed $\mathcal{S}\left(\mathbb{R}^{2N}\right)$ is the projective completion of the above algebraic tensor product with respect to seminorms r_k given by (5.87). From the seminorms (r_k) given by (5.41) it turns out that $\mathcal{S}\left(\mathbb{R}^{2N}\right)$ is a space of $\mathbb{C}$-linear combinations

$$\sum_{\nu \in I} c_\nu f_\nu;\ \text{where } c_\nu = c_{\left(\left(m^\nu_1, n^\nu_1\right), \dots, \left(m^\nu_N, n^\nu_N\right)\right)} \in \mathbb{C}$$

such that for any $k = (k_1, \dots, k_N) \in \left(\mathbb{N}^0\right)^N$ following condition holds

$$r_k\left(\sum_{\nu \in I} c_\nu f_\nu\right) = \left(\theta^{2(k_1 + \dots + k_N)} \sum_{\nu \in I} |c_\nu|^2 \prod_{p=1}^N \left(m^\nu_p + \tfrac{1}{2}\right)^{k_p} \left(n^\nu_p + \tfrac{1}{2}\right)^{k_p}\right)^{1/2} < \infty. \qquad (5.87)$$

If $M \in \mathbb{N}$ and and $I_M \subset I$ is a finite subset such that

$$I_M = \left\{\left(\left(m^\nu_1, n^\nu_1\right), \dots, \left(m^\nu_N, n^\nu_N\right)\right) \in I \mid m^\nu_1, n^\nu_1, \dots, m^\nu_N, n^\nu_N \le M\right\} \qquad (5.88)$$

then (5.87) is equivalent to

$$\sum_{\nu \in I \setminus I_M} |c_\nu| \prod_{p=1}^N \left(m^\nu_p + \tfrac{1}{2}\right)^{k_p} \left(n^\nu_p + \tfrac{1}{2}\right)^{k_p} < \infty.$$

From the above equation and $(m + n + 1)^k < 2^k \left(m + \frac{1}{2}\right)^k \left(n + \frac{1}{2}\right)^k, \forall m, n, k > 1$. it follows that for any $M > 1$ and $l > 1$ following condition holds

$$\sum_{\nu \in I \setminus I_M} |c_\nu| \prod_{p=1}^N \left(m^\nu_p + n^\nu_p + 1\right)^l < \infty. \qquad (5.89)$$

From the Lemma 5.29 it turns out that the system of seminorms $\|\cdot\|_s$ given by (3.11) is equivalent to the system of seminorms $\|\cdot\|_{(t_1,\dots,t_{2N})}$ given by

$$\|\widetilde{a}\|_{(t_1,\dots,t_{2N})} \stackrel{\text{def}}{=} \left\|\frac{\partial^{t_1+\dots+t_{2N}}}{\partial x_1^{t_1}\dots\partial x_{2N}^{t_{2N}}}\widetilde{a}\right\|_{\text{op}}$$

This lemma is true if for any $(t_1,\dots,t_{2N}) \in \left(\mathbb{N}^0\right)^{2N}$ from

$$\left\|\frac{\partial^{t_1+\dots+t_{2N}}}{\partial x_1^{t_1}\dots\partial x_{2N}^{t_{2N}}}\widetilde{a}\right\|_{\text{op}} < \infty$$

it follows that for any $\varepsilon > 0$ there is a finite subset $I_f \subset I$ such that

$$\sum_{\nu\in I\setminus I_f} |c_\nu| \left\|\frac{\partial^{t_1+\dots+t_{2N}}}{\partial x_1^{t_1}\dots\partial x_{2N}^{t_{2N}}} f_\nu\right\|_{\text{op}} < \varepsilon. \tag{5.90}$$

Let us replace coordinates $x_1,\dots,x_{2N}$ with coordinates $p_1,q_1,\dots,p_N,q_N$ such that p_j,q_j are coordinates on j^{th} term of the product $\underbrace{\mathcal{S}\left(\mathbb{R}^2\right)\otimes\dots\otimes\mathcal{S}\left(\mathbb{R}^2\right)}_{N-\text{times}} \subset \mathcal{S}\left(\mathbb{R}^{2N}\right)$. From the equation

$$\frac{\partial^{t_1+\dots+t_{2N}}}{\partial x_1^{t_1}\dots\partial x_{2N}^{t_{2N}}} f_\nu = \frac{\partial^{t_1,t_2}}{\partial p_1^{t_1}\partial q_1^{t_2}} f_{m_1^\nu,n_1^\nu} \otimes\dots\otimes \frac{\partial^{t_{2N-1},t_{2N}}}{\partial p_N^{t_1}\partial q_N^{t_2}} f_{m_N^\nu,n_N^\nu} \tag{5.91}$$

it follows that

$$\left\|\frac{\partial^{t_1+\dots+t_{2N}}}{\partial x_1^{t_1}\dots\partial x_{2N}^{t_{2N}}} f_\nu\right\|_{\text{op}} = \left\|\frac{\partial^{t_1,t_2}}{\partial p_1^{t_1}\partial q_1^{t_2}} f_{m_1^\nu,n_1^\nu}\right\|_{\text{op}} \cdot\ldots\cdot \left\|\frac{\partial^{t_{2N-1},t_{2N}}}{\partial p_N^{t_1}\partial q_N^{t_2}} f_{m_N^\nu,n_N^\nu}\right\|_{\text{op}}. \tag{5.92}$$

Our proof can be simplified if we use scaling construction 5.20, i.e. we set $\theta = 2$. Indeed many quantitative results does not depend on θ. Similarly to (5.49) we write $f\times g$ instead of $f\star_2 g$. Moreover one can use given by (5.54) coordinates $a,\overline{a}$ instead of p,q. From (5.55) it follows that

$$\frac{\partial f}{\partial a} = -\overline{a}\times f + f\times\overline{a},\quad \frac{\partial f}{\partial \overline{a}} = a\times f - f\times a,$$

and taking into account (5.57) one has

$$\begin{aligned}\frac{\partial f_{mn}}{\partial a} &= -\sqrt{2m+2}f_{m+1,n} + \sqrt{2n}f_{m,n-1},\\ \frac{\partial f_{mn}}{\partial \overline{a}} &= \sqrt{2m}f_{m-1,n} - \sqrt{2n+2}f_{m,n+1}.\end{aligned}$$

If $t_1,t_2 \in \mathbb{N}^0$ and $|t| = t_1+t_2$ then from the above equations and $\|f_{mn}\|_{\text{op}} = 1$ it follows that

$$\begin{gathered}\left\|\frac{\partial^{t_1,t_2}}{\partial a^{t_1}\partial \overline{a}^{t_2}} f_{mn}\right\|_{\text{op}} \le \\ \le \left(\sqrt{2m+2+|t|}+\sqrt{2n+|t|}\right)^{t_1}\left(\sqrt{2m+|t|}+\sqrt{2n+2+|t|}\right)^{t_2} < \\ < (2m+2n+2+2\,|t|)^{t_1}\,(2m+2n+2+2\,|t|)^{t_2} = (2m+2n+2+2\,|t|)^{|t|}.\end{gathered} \tag{5.93}$$

If $m,n \ge M$ then

$$(2m+2n+2+2\,|t|)^{|t|} \le \left(\frac{4M+2+2\,|t|}{4M+2}\right)^{|t|}(2m+2n+2)^{|t|} \tag{5.94}$$

Let us consider a differential operator

$$P = \frac{\partial^{t_1+\dots+t_{2N}}}{\partial a_1^{t_1}\partial \overline{a}_1^{t_2}\dots\partial a_N^{t_{2N-1}}\partial \overline{a}_N^{t_{2N}}} = P_1\otimes\dots\otimes P_N = \frac{\partial^{t_1,t_2}}{\partial a_1^{t_1}\partial \overline{a}_1^{t_2}}\otimes\dots\otimes\frac{\partial^{t_{2N-1},t_{2N}}}{\partial a_N^{t_{2N-1}}\partial \overline{a}_N^{t_{2N}}}, \tag{5.95}$$

and denote by $|t| = t_1+\dots+t_{2N}$. If $M\in\mathbb{N}$ and I_M is given by (5.88) then from (5.94) it turns out

$$\begin{gathered}\sum_{\nu\in I\setminus I_M}|c_\nu|\prod_{p=1}^{N}\left(2m_p^\nu+2n_p^\nu+2+2\,|t|\right)^{|t|} < \\ < \left(\frac{4M+2+2\,|t|}{4M+2}\right)^{|t|}\sum_{\nu\in I\setminus I_M}|c_\nu|\prod_{p=1}^{N}\left(2m_p^\nu+2n_p^\nu+2\right)^{|t|}.\end{gathered}$$

From (5.89) it follows that right part of the above equation is convergent, hence one has

$$\sum_{\nu\in I}|c_\nu|\prod_{p=1}^{N}\left(2m_p^\nu+2n_p^\nu+2+2\,|t|\right)^{|t|} < \infty \tag{5.96}$$

From (5.93) and (5.96) it follows that the series

$$\sum_{\nu\in I} c_\nu P f_\nu$$

is $\|\cdot\|_{\text{op}}$-norm convergent (cf. (5.35)) , so the series $\sum_{\nu\in I} c_\nu f_\nu$ is convergent with respect to seminorms $\|\cdot\|_s$ given by (3.11).

□

5.6.5 Covering of spectral triple

Following theorem completely describes infinite coverings of noncommutative tori.

Theorem 5.35. *Let $\mathfrak{S}_{\left(C^\infty(\mathbb{T}_\theta), L^2\left(C\left(\mathbb{T}^{2N}_\theta\right),\tau\right)\otimes\mathbb{C}^{2^N}, D\right)} \in \mathfrak{CohTriple}$ be a coherent sequence of spectral triples given by* (5.62). *Let $\widehat{\pi}^\oplus : \widehat{C\left(\mathbb{T}^{2N}_\theta\right)} \to \bigoplus_{g\in J} gL^2\left(\mathbb{R}^{2N}_\theta\right)$ be an equivariant representation given by* (5.61). *Following condition holds:*

(i) If $C_0^\infty\left(\mathbb{R}^{2N}_\theta\right)$ is the smooth algebra of $\mathfrak{S}_{\left(C^\infty(\mathbb{T}_\theta), L^2\left(C\left(\mathbb{T}^{2N}_\theta\right),\tau\right)\otimes\mathbb{C}^{2^N}, D\right)}$ with respect to $\widehat{\pi}^\oplus$ then $C_0^\infty\left(\mathbb{R}^{2N}_\theta\right)$ is the completion of $\mathcal{S}\left(\mathbb{R}^{2N}_\theta\right)$ with respect to seminorms given by (5.78),

(ii) The sequence $\mathfrak{S}_{\left(C^\infty(\mathbb{T}_\theta), L^2\left(C\left(\mathbb{T}^{2N}_\theta\right),\tau\right)\otimes\mathbb{C}^{2^N}, D\right)}$ is regular with respect to $\widehat{\pi}^\oplus$,

(iii) If $\widetilde{D}$ is given by (5.77) *then the triple*

$$\left(C_0^\infty\left(\mathbb{R}^{2N}_\theta\right), L^2\left(\mathbb{R}^{2N}\right)\otimes\mathbb{C}^{2^N}, \widetilde{D}\right)$$

is the $\left(C\left(\mathbb{T}^{2N}_\theta\right), C_0\left(\mathbb{R}^{2N}_\theta\right), \mathbb{Z}^{2N}\right)$-lift of $\left(C\left(\mathbb{T}^{2N}_\theta\right), L^2\left(\mathbb{T}^{2N}\right)\otimes\mathbb{C}^{2^N}, D\right)$.

Proof. (i) From the Lemma 5.34 it follows that $C_0^\infty\left(\mathbb{R}^{2N}_\theta\right)$ is contained in the completion of $\mathcal{S}\left(\mathbb{R}^{2N}_\theta\right)$ with respect to seminorms given by (5.78). From the Lemma 5.33 it turns out $\mathcal{S}\left(\mathbb{R}^{2N}_\theta\right) \subset C_0^\infty\left(\mathbb{R}^{2N}_\theta\right)$.
(ii) The algebra $\mathcal{S}\left(\mathbb{R}^{2N}_\theta\right)$ is dense in $C_0\left(\mathbb{R}^{2N}_\theta\right)$, so $C_0^\infty\left(\mathbb{R}^{2N}_\theta\right)$ is dense in $C_0\left(\mathbb{R}^{2N}_\theta\right)$.
(iii) Follows from the construction 5.32. □

6 Isospectral deformations and their coverings

A very general construction of isospectral deformations of noncommutative geometries is described in [6]. The construction implies in particular that any compact Spin-manifold M whose isometry group has rank ≥ 2 admits a natural one-parameter isospectral deformation to noncommutative geometries M_θ. We let $\left(C^\infty(M), \mathcal{H} = L^2(M,S), \not{D}\right)$ be the canonical spectral triple associated with a compact spin-manifold M. We recall that $\mathcal{A} = C^\infty(M)$ is the algebra of smooth functions on M, S is the spinor bundle and $\not{D}$ is the Dirac operator. Let us assume that the group $\mathrm{Isom}(M)$ of isometries of M has rank $r \geq 2$. Then, we have an inclusion

$$\mathbb{T}^2 \subset \mathrm{Isom}(M),$$

with $\mathbb{T}^2 = \mathbb{R}^2/2\pi\mathbb{Z}^2$ the usual torus, and we let $U(s), s \in \mathbb{T}^2$, be the corresponding unitary operators in $\mathcal{H} = L^2(M,S)$ so that by construction

$$U(s)\,\not{D} = \not{D}\,U(s).$$

Also,

$$U(s)\,a\,U(s)^{-1} = \alpha_s(a), \ \ \forall\, a \in \mathcal{A}, \tag{6.1}$$

where $\alpha_s \in \mathrm{Aut}(\mathcal{A})$ is the action by isometries on the algebra of functions on M.
We let $p = (p_1, p_2)$ be the generator of the two-parameters group $U(s)$ so that

$$U(s) = \exp(i(s_1 p_1 + s_2 p_2)).$$

The operators p_1 and p_2 commute with D. Both p_1 and p_2 have integral spectrum,

$$\mathrm{Spec}(p_j) \subset \mathbb{Z}\,,\ \ j = 1, 2\,.$$

One defines a bigrading of the algebra of bounded operators in $\mathcal{H}$ with the operator T declared to be of bidegree (n_1, n_2) when,

$$\alpha_s(T) = \exp(i(s_1 n_1 + s_2 n_2))\, T\,,\ \ \forall\, s \in \mathbb{T}^2\,,$$

where $\alpha_s(T) = U(s)\, T\, U(s)^{-1}$ as in (6.1).

Any operator T of class C^∞ relative to α_s (i. e. such that the map $s \to \alpha_s(T)$ is of class C^∞ for the norm topology) can be uniquely written as a doubly infinite norm convergent sum of homogeneous elements,

$$T = \sum_{n_1, n_2} \widehat{T}_{n_1, n_2}\,,$$

with $\widehat{T}_{n_1, n_2}$ of bidegree (n_1, n_2) and where the sequence of norms $||\widehat{T}_{n_1, n_2}||$ is of rapid decay in (n_1, n_2). Let $\lambda = \exp(2\pi i\theta)$. For any operator T in $\mathcal{H}$ of class C^∞ we define its left twist $l(T)$ by

$$l(T) = \sum_{n_1, n_2} \widehat{T}_{n_1, n_2}\, \lambda^{n_2 p_1}\,, \tag{6.2}$$

and its right twist $r(T)$ by

$$r(T) = \sum_{n_1, n_2} \widehat{T}_{n_1, n_2}\, \lambda^{n_1 p_2}\,,$$

Since $|\lambda| = 1$ and p_1, p_2 are self-adjoint, both series converge in norm. Denote by $C^\infty\,(M)_{n_1, n_2} \subset C^\infty\,(M)$ the $\mathbb{C}$-linear subspace of elements of bidegree (n_1, n_2).
One has,

Lemma 6.1. *[6]*

a) *Let x be a homogeneous operator of bidegree (n_1, n_2) and y be a homogeneous operator of bidegree (n'_1, n'_2). Then,*

$$l(x)\, r(y)\, -\, r(y)\, l(x) = (x\, y\, - y\, x)\, \lambda^{n'_1 n_2} \lambda^{n_2 p_1 + n'_1 p_2} \tag{6.3}$$

In particular, $[l(x), r(y)] = 0$ if $[x, y] = 0$.

b) *Let x and y be homogeneous operators as before and define*

$$x * y = \lambda^{n'_1 n_2}\, xy\,; \tag{6.4}$$

then $l(x)l(y) = l(x * y)$.

The product $*$ defined in (6.4) extends by linearity to an associative product on the linear space of smooth operators and could be called a $*$-product. One could also define a deformed 'right product'. If x is homogeneous of bidegree (n_1, n_2) and y is homogeneous of bidegree (n'_1, n'_2) the product is defined by

$$x *_r y = \lambda^{n_1 n'_2} xy .$$

Then, along the lines of the previous lemma one shows that $r(x)r(y) = r(x *_r y)$.
We can now define a new spectral triple where both $\mathcal{H}$ and the operator D are unchanged while the algebra $C^\infty(M)$ is modified to $l(C^\infty(M))$. By Lemma 6.1 b) one checks that $l(C^\infty(M))$ is still an algebra. Since D is of bidegree $(0,0)$ one has,

$$[D, l(a)] = l([D, a])$$

which is enough to check that $[D, x]$ is bounded for any $x \in l(\mathcal{A})$. There is a spectral triple $(l(C^\infty(M)), \mathcal{H}, D)$.

Denote by $C(M_\theta)$ the operator norm completion (equivalently C^*-norm completion) of $l(C^\infty(M))$, and denote by $\rho : C(M) \to L^2(M, S)$ (resp. $\pi_\theta : C(M_\theta) \to B\left(L^2(M, S)\right)$) natural representations.

6.1 Finite-fold coverings

6.1.1 Basic construction

Let M be a spin - manifold with the smooth action of $\mathbb{T}^2$. Let $\pi : \widetilde{M} \to M$ be a finite-fold covering. Let $\widetilde{x}_0 \in \widetilde{M}$ and $x_0 = \pi(\widetilde{x}_0)$. Denote by $\varphi : \mathbb{R}^2 \to \mathbb{R}^2/\mathbb{Z}^2 = \mathbb{T}^2$ the natural covering. There are two closed paths $\omega_1, \omega_2 : [0,1] \to M$ given by

$$\omega_1(t) = \varphi(t, 0)\, x_0, \ \omega_2(t) = \varphi(0, t)\, x_0.$$

There are lifts of these paths, i.e. maps $\widetilde{\omega}_1, \widetilde{\omega}_2 : [0,1] \to \widetilde{M}$ such that

$$\begin{aligned} \widetilde{\omega}_1(0) = \widetilde{\omega}_2(0) = \widetilde{x}_0, \\ \pi(\widetilde{\omega}_1(t)) = \omega_1(t), \\ \pi(\widetilde{\omega}_2(t)) = \omega_2(t). \end{aligned}$$

Since π is a finite-fold covering there are $N_1, N_2 \in \mathbb{N}$ such that if

$$\gamma_1(t) = \varphi(N_1 t, 0)\, x_0, \ \gamma_2(t) = \varphi(0, N_2 t)\, x_0.$$

and $\widetilde{\gamma}_1$ (resp. $\widetilde{\gamma}_2$) is the lift of γ_1 (resp. γ_2) then both $\widetilde{\gamma}_1$, $\widetilde{\gamma}_2$ are closed. Let us select minimal values of N_1, N_2. If $\mathrm{pr}_n : S^1 \to S^1$ is an n listed covering and $\mathrm{pr}_{N_1, N2}$ the covering given by

$$\widetilde{\mathbb{T}}^2 = S^1 \times S^1 \xrightarrow{\mathrm{pr}_{N_1} \times \mathrm{pr}_{N_2}} S^1 \times S^1 = \mathbb{T}^2$$

then there is the action $\widetilde{\mathbb{T}}^2 \times \widetilde{M} \to \widetilde{M}$ such that

$$\begin{array}{ccc}\widetilde{\mathbb{T}}^2 \times \widetilde{M} & \longrightarrow & \widetilde{M} \\ \Big\downarrow \mathrm{pr}_{N_1N_2} \times \pi & & \Big\downarrow \pi \\ \mathbb{T}^2. \times M & \longrightarrow & M \end{array}$$

where $\widetilde{\mathbb{T}}^2 \approx \mathbb{T}^2$. Let $\widetilde{p} = (\widetilde{p}_1, \widetilde{p}_2)$ be the generator of the associated with $\widetilde{\mathbb{T}}^2$ two-parameters group $\widetilde{U}(s)$ so that

$$\widetilde{U}(s) = \exp\left(i\left(s_1\widetilde{p}_1 + s_2\widetilde{p}_2\right)\right).$$

The covering $\widetilde{M} \to M$ induces an involutive injective homomorphism

$$\varphi : C^\infty(M) \to C^\infty\left(\widetilde{M}\right).$$

Since $\widetilde{M} \to M$ is a covering $C^\infty\left(\widetilde{M}\right)$ is a finitely generated projective $C^\infty(M)$-module, i.e. there is the following direct sum of $C^\infty\left(\widetilde{M}\right)$-modules

$$C^\infty\left(\widetilde{M}\right) \bigoplus P = C^\infty(M)^n \tag{6.5}$$

such that

$$\varphi\left(C^\infty(M)\right)_{n_1,n_2} \subset C^\infty\left(\widetilde{M}\right)_{n_1N_1,\ n_2N_2}.$$

Let $\theta, \widetilde{\theta} \in \mathbb{R}$ be such that

$$\widetilde{\theta} = \frac{\theta + n}{N_1N_2}, \text{ where } n \in \mathbb{Z}.$$

If $\lambda = e^{2\pi i\theta}$, $\widetilde{\lambda} = e^{2\pi i\widetilde{\theta}}$ then $\lambda = \widetilde{\lambda}^{N_1N_2}$. There are isospectral deformations $C^\infty(M_\theta), C^\infty\left(\widetilde{M}_{\widetilde{\theta}}\right)$ and $\mathbb{C}$-linear isomorphisms $l : C^\infty(M) \to C^\infty(M_\theta)$, $\widetilde{l} : C^\infty\left(\widetilde{M}\right) \to C^\infty\left(\widetilde{M}_{\widetilde{\theta}}\right)$. These isomorphisms and the inclusion φ induce the inclusion

$$\varphi_\theta : C^\infty(M_\theta) \to C^\infty\left(\widetilde{M}_{\widetilde{\theta}}\right),$$
$$\varphi_{\widetilde{\theta}}\left(C^\infty(M_\theta)\right)_{n_1,n_2} \subset C^\infty\left(\widetilde{M}_{\widetilde{\theta}}\right)_{n_1N_1,\ n_2N_2}.$$

Theorem 6.2. *[12] The triple* $\left(C(M_\theta), C\left(\widetilde{M}_{\widetilde{\theta}}\right), G\left(\widetilde{M} \mid M\right)\right)$ *is an unital noncommutative finite-fold covering.*

6.1.2 Induced representations and finite-fold coverings of spectral triples

Following facts are evident:

- If both $\rho : C(M_\theta) \to B\left(L^2(M,S)\right)$ and $\widetilde{\rho} : C\left(\widetilde{M}_{\widetilde{\theta}}\right) \to B\left(L^2\left(\widetilde{M}, \widetilde{S}\right)\right)$ are natural representations then $\widetilde{\rho}$ is induced by $\left(\rho, \left(C(M_\theta), C\left(\widetilde{M}_{\widetilde{\theta}}\right), G\left(\widetilde{M}, M\right)\right)\right)$,

- If the spectral triple $\left(C^{\infty}\left(\widetilde{M}\right), L^2\left(\widetilde{M}, \widetilde{S}\right), \widetilde{\not{D}}\right)$ is a $\left(C\left(M\right), C\left(\widetilde{M}\right), G\left(\widetilde{M} \mid M\right)\right)$-lift of $\left(C^{\infty}\left(M\right), L^2\left(M, S\right), \not{D}\right)$ then the noncommutative spectral triple

$$\left(\widetilde{I}C^{\infty}\left(\widetilde{M}\right), L^2\left(\widetilde{M}, \widetilde{S}\right), \widetilde{\not{D}}\right)$$

is a $\left(C\left(M_{\theta}\right), C\left(\widetilde{M}_{\widetilde{\theta}}\right), G\left(\widetilde{M} \mid M\right)\right)$-lift of $\left(IC^{\infty}\left(M\right), L^2\left(M, S\right), \not{D}\right)$.

6.2 Infinite coverings

Let $\mathfrak{S}_M = \left\{M = M^0 \leftarrow M^1 \leftarrow ... \leftarrow M^n \leftarrow ...\right\} \in \mathfrak{FinTop}$ be an infinite sequence of spin - manifolds and regular finite-fold covering. Suppose that there is the action $\mathbb{T}^2 \times M \to M$ given by (6.1). From the Theorem 6.2 it follows that there is the algebraical finite covering sequence

$$\mathfrak{S}_{C(M_{\theta})} = \left\{C\left(M_{\theta}\right) \to ... \to C\left(M^n_{\theta_n}\right) \to ...\right\}.$$

So one can calculate a finite noncommutative limit of the above sequence. This article does not contain detailed properties of this noncommutative limit, because it is not known yet by the author of this article.

Acknowledgment

I am very grateful to Prof. Joseph C Varilly and Arup Kumar Pal for advising me on the properties of Moyal planes resp. equivariant spectral triples.

References

[1] M. Auslander; I. Reiten; S.O. Smalø. *Galois actions on rings and finite Galois coverings*. Mathematica Scandinavica (1989), Volume: 65, Issue: 1, page 5-32, ISSN: 0025-5521; 1903-1807/e , 1989.

[2] W. Arveson. *An Invitation to C^*-Algebras*, Springer-Verlag. ISBN 0-387-90176-0, 1981.

[3] B. Blackadar. *K-theory for Operator Algebras*, Second edition. Cambridge University Press 1998.

[4] F. Brickell and R. S. Clark. *Differentiable manifolds; An introduction.* London ; New York: V. N. Reinhold Co., 1970.

[5] Alain Connes. *Noncommutative Geometry*, Academic Press, San Diego, CA, 661 p., ISBN 0-12-185860-X, 1994.

[6] Alain Connes, Giovanni Landi. *Noncommutative Manifolds the Instanton Algebra and Isospectral Deformations*, arXiv:math/0011194, 2001.

[7] José M. Gracia-Bondía, Joseph C. Várilly. *Algebras of Distributions suitable for phase-space quantum mechanics. I*. Escuela de Matemática, Universidad de Costa Rica, San José, Costa Rica J. Math. Phys 29 (1988), 869-879, 1988.

[8] J. C. Várilly and J. M. Gracia-Bondía, *Algebras of distributions suitable for phase-space quantum mechanics II: Topologies on the Moyal algebra*, J. Math. Phys. **29** (1988), 880–887. 1988.

[9] Manfredo P. do Carmo. *Riemannian Geometry.* Birkhäuser, 1992.

[10] V. Gayral, J. M. Gracia-Bondía, B. Iochum, T. Schücker, J. C. Varilly *Moyal Planes are Spectral Triples*. arXiv:hep-th/0307241, 2003.

[11] *Lecture notes on noncommutative geometry and quantum groups*, Edited by Piotr M. Hajac.

[12] Petr Ivankov. *Quantization of noncompact coverings*, arXiv:1702.07918, 2017.

[13] M. Karoubi. *K-theory, An Introduction.* Springer-Verlag 1978.

[14] S. Kobayashi, K. Nomizu. *Foundations of Differential Geometry*. Volume 1. Interscience publishers a division of John Willey & Sons, New York - London. 1963.

[15] Bram Mesland. *Unbounded bivariant K-theory and correspondences in noncommutative geometry* arXiv:0904.4383, 2009

[16] James R. Munkres. *Topology.* Prentice Hall, Incorporated, 2000.

[17] G.J. Murphy. *C^*-Algebras and Operator Theory.* Academic Press 1990.

[18] Alexander Pavlov, Evgenij Troitsky. *Quantization of branched coverings.* Russ. J. Math. Phys. (2011) 18: 338. doi:10.1134/S1061920811030071, 2011.

[19] Pedersen G.K. *C^*-algebras and their automorphism groups*. London ; New York : Academic Press, 1979.

[20] E.H. Spanier. *Algebraic Topology.* McGraw-Hill. New York 1966.

[21] J.C. Várilly. *An Introduction to Noncommutative Geometry*. EMS 2006.